

Well-posedness of semilinear stochastic wave equations with Hölder continuous coefficients

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Abstract

We prove that semilinear stochastic abstract wave equations, including wave and plate equations, are well-posed in the strong sense with an α -Hölder continuous drift coefficient, if $\alpha \in (2/3, 1)$. The uniqueness may fail for the corresponding deterministic PDE and well-posedness is restored by adding an external random forcing of white noise type. This shows a kind of regularization by noise for the semilinear wave equation. To prove the result we introduce an approach based on backward stochastic differential equations. We also establish regularizing properties of the transition semigroup associated to the stochastic wave equation by using control theoretic results.

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1 Introduction

We prove well-posedness in the strong sense for semilinear stochastic abstract wave equations, including wave and plate equations. Let us consider the following non-linear stochastic wave equation with Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial^2}{\partial \tau^2} y(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + b(\tau, \xi, y(\tau, \xi)) + \dot{W}(\tau, \xi), & \xi \in (0, 1), \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \\ \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), & \tau \in (0, T], \quad \xi \in [0, 1], \end{cases} \quad (1.1)$$

where $x_0 \in H_0^1([0, 1])$, $x_1 \in L^2([0, 1])$ and $\dot{W}(\tau, \xi)$ is a space-time white noise on $[0, T] \times [0, 1]$ which describes an external random forcing; we treat it as a time-derivative of a cylindrical Wiener process with values in $L^2([0, 1])$. Moreover b is a bounded measurable function which is Hölder continuous of exponent $\alpha \in (2/3, 1)$ with respect to the y -variable; see Hypothesis 3.1 for the more general assumptions. To get pathwise uniqueness for (1.1) (see Theorem 6.3) we introduce an approach based on backward stochastic differential equations.

Without the noise $\dot{W}(\tau, \xi)$ the corresponding nonlinear deterministic equation is in general not well-posed; see Section 3.3. Thus our result is a kind of regularization by additive noise for semilinear stochastic wave equations. There are already results in this direction at the level of SPDEs of parabolic type (see

[21], [7], [8], [27], [9], [33] and the references therein). For related results on well-posedness of SPDEs by a kind of multiplicative noise perturbations, see [15], [12], [14], [13] and the references therein. Coming into the details of the problem we treat in the present paper, indeed we study general abstract wave equations of the form

$$\begin{cases} \frac{d^2 y}{d\tau^2}(\tau) = \Lambda y(\tau) + B(t, y(\tau), \frac{dy}{d\tau}(\tau)) + \dot{W}(\tau), \\ y(0) = x_0, \\ \frac{dy}{d\tau}(0) = x_1, \quad \tau \in (0, T], \end{cases} \quad (1.2)$$

where $\Lambda : \mathcal{D}(\Lambda) \subset U \rightarrow U$ is a positive self-adjoint operator on a separable Hilbert space U (see, for instance, Example 5.8 and Section 5.5.2 in [11], [3] and the references therein) and $\{W(\tau) = W_\tau, \tau \geq 0\}$ is a cylindrical Wiener process with values in U . Many linear stochastic equations modelling the vibrations of elastic structures can be written in the form (1.2) with $B = 0$ where y stands for the displacement field (for instance, we consider the stochastic plate equation in Section 3.2).

Comparing with (1.1), we have that $\Lambda = -\frac{d^2}{dx^2}$ with Dirichlet boundary conditions,

$$U = L^2([0, 1]), \quad \mathcal{D}(\Lambda) = H_0^1([0, 1]) \cap H^2([0, 1]), \quad \mathcal{D}(\Lambda^{1/2}) = H_0^1([0, 1]) \quad (1.3)$$

and $\mathcal{D}(\Lambda^{-1/2}) = H^{-1}([0, 1])$. Solutions to equations (1.2) do not evolve in the usual space $K = \mathcal{D}(\Lambda^{1/2}) \times U$ even if $B = 0$ but in the larger space $H = U \times \mathcal{D}(\Lambda^{-1/2})$ (see also the comments on formula (2.6)).

The existence of a weak solution $X_\tau^{0,x} = (y(\tau), \frac{dy}{d\tau}(\tau))$ to (1.2) with values in H and with continuous paths is well known, for any $x = (x_0, x_1) \in H$; see Section 2 for more details. It follows by the Girsanov theorem (cf. [11], [28], [26] and Remark 2.1) writing (1.2) as

$$dX_\tau^{0,x} = AX_\tau^{0,x} d\tau + GB(\tau, X_\tau^{0,x}) d\tau + GdW_\tau, \quad \tau \in [0, T], \quad X_0^{0,x} = x \in H, \quad (1.4)$$

where A is the generator of the wave group in H and $GdW_\tau = \begin{pmatrix} 0 \\ dW_\tau \end{pmatrix}$. We require that $B : [0, T] \times H \rightarrow U$ is Borel, bounded and α -Hölder continuous in the x -variable, $\alpha \in (2/3, 1)$ (cf. Hypothesis 2).

To prove pathwise uniqueness for (1.4), we first investigate regularizing properties of the H -valued Ornstein-Uhlenbeck semigroup (R_t) (see Section 4). We have $R_\tau[\Phi](x) = \mathbb{E}[\Phi(X_\tau^{0,x})]$, $\tau \geq 0$, $\Phi \in B_b(H, H)$, where $X_\tau^{0,x}$ is the Ornstein-Uhlenbeck process solving (1.4) when $B = 0$. Proceeding as in [10], [25] and [7] one can first prove Gâteaux differentiability of $R_\tau[\Phi]$, $\tau > 0$. Then using sharp results on the behaviour of the minimal energy for the linear controlled system

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = h \in H, \end{cases} \quad (1.5)$$

with controls $u \in L_{loc}^2(0, \infty; \mathcal{D}(\Lambda^{-1/2}))$ (see Theorem 3 in [1] and Theorem A.1 in Appendix) we are able to prove new regularity results for the derivative of $R_\tau[\Phi]$ in the directions of the noise $Ga = \begin{pmatrix} 0 \\ a \end{pmatrix}$, $a \in U$. In particular we show that such derivative $\nabla^G R_\tau[\Phi](x)$ belongs to the space $L_2(U, H)$ of Hilbert-Schmidt operators from U into H (see Lemmas 4.2 and 4.4).

In Section 5 we introduce backward stochastic differential equations (BSDEs from now on) for the unknown pair of processes $(Y^{t,x}, Z^{t,x})$, coupled with the Ornstein-Uhlenbeck process $\Xi^{t,x}$ starting from x at time t :

$$\begin{cases} d\Xi_\tau^{t,x} = A\Xi_\tau^{t,x} d\tau + GdW_\tau, & \tau \in [t, T], \\ \Xi_t^{t,x} = x, \\ -dY_\tau^{t,x} = -AY_\tau^{t,x} d\tau + GB(\tau, \Xi_\tau^{t,x}) d\tau + Z_\tau^{t,x} B(\tau, \Xi_\tau^{t,x}) d\tau - Z_\tau^{t,x} dW_\tau, & \tau \in [t, T], \\ Y_T^{t,x} = 0. \end{cases} \quad (1.6)$$

The process $Y^{t,x}$ takes values in H and $Z^{t,x}$ in the space $L_2(U, H)$ (cf. [22], [4] and [17]). We study first differentiability of $(Y^{t,x}, Z^{t,x})$ with respect to x assuming in addition that the coefficient B is differentiable. Such type of results, together with the identification of $Z^{t,x}$ with the directional derivative of $Y^{t,x}$, are known also in the infinite dimensional case when $Y^{t,x}$ is real, see [17]; here we extend these results to the case when $Y^{t,x}$ is Hilbert space valued (see Proposition 5.1 and Lemma 5.3). Then, using

the results of Section 4 and an approximation argument, we are able to study regularity properties of solutions $(Y^{t,x}, Z^{t,x})$ together with the identification of $Z^{t,x}$ in the case of an Hölder continuous drift B (see Theorem 5.4 which holds under more general assumptions on B and also Lemma 5.5).

These results allow to get in Section 6 the important identity

$$X_\tau^{0,x} = e^{\tau A}x + e^{\tau A}v(0, x) - v(\tau, X_\tau^x) + \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) dW_s + \int_0^\tau e^{(\tau-s)A} G dW_s \quad (1.7)$$

involving a “regular function” v related to $Y^{t,x}$ ($v(t, x) = Y_t^{t,x}$, $(t, x) \in [0, T] \times H$; see (5.7)). Note that the irregular coefficient B is not present in (1.7). This identity allows to prove pathwise uniqueness and Lipschitz dependence from the initial conditions of the solution $X_\tau^{0,x}$ (see Theorem 6.3). Identities like (1.7) are established in [15], [7], [8], [9], [33] by the so-called Ito-Tanaka trick which is a variant of the Zvonkin method used in [32] (see also our Remark 6.2 and [16]). Here we prove (1.7) by using the mild form of the BSDE, which, together with the group property of A , allows to remove the “bad term” B of the semilinear stochastic wave equation.

2 Notations and preliminary results

Given two real separable Hilbert spaces H and K we denote by $L(H, K)$ the space of bounded linear operators from H to K , endowed with the usual operator norm; $L_2(H, K)$ is the subspace of all Hilbert-Schmidt operators endowed with the Hilbert-Schmidt norm $\|\cdot\|_{L_2(H, K)}$. Let E be a Banach space. $B_b(H, E)$ is the space of all Borel and bounded functions from H into E endowed with the supremum norm $\|\cdot\|_\infty$, $\|f\|_\infty = \sup_{x \in H} |f(x)|_E$, $f \in B_b(H, E)$. $C_b(H, E)$ is its subspace consisting of all uniformly continuous and bounded functions from H into E . The space $C_b^1(H, E)$ is the space of all functions in $C_b(H, E)$ which are Fréchet differentiable on H with bounded and uniformly continuous Fréchet derivative $\nabla f : H \rightarrow L(H, E)$; it is a Banach space endowed with the norm $\|\cdot\|_{C_b^1}$, $\|f\|_{C_b^1} = \|f\|_\infty + \|\nabla f\|_\infty$, $f \in C_b^1(H, E)$. Moreover, $C_b^\infty(H, E)$ is the space of all functions in $C_b(H, E)$ which are infinitely many times Fréchet differentiable with bounded Fréchet derivatives of any order. By $C([0, T] \times H, E)$ we denote the space of continuous functions from the product space $[0, T] \times H$ into E . Moreover, $B_b([0, T] \times H, E)$ is the Banach space of bounded Borel measurable functions from $[0, T] \times H$ into E endowed with the sup norm.

We also introduce, for $0 < \alpha < 1$, the space $C_b^\alpha(H, E)$ of all functions in $C_b(H, E)$ which are also α -Hölder continuous, and we endow $C_b^\alpha(H, E)$ with the usual Hölder norm $\|\cdot\|_\alpha$. When $E = \mathbb{R}$, we set $C_b^\alpha(H, \mathbb{R}) = C_b^\alpha(H)$; recall the following result from interpolation theory (see Theorem 2.3.3 in [10]):

$$(C_b(H), C_b^1(H))_{\alpha/2, \infty} = C_b^\alpha(H). \quad (2.1)$$

Let U be a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_U$ and norm $|\cdot|_U$. To study (1.2) we assume that

Hypothesis 1. $\Lambda : \mathcal{D}(\Lambda) \subset U \rightarrow U$ is a given positive self-adjoint operator and there exists Λ^{-1} which is a trace class operator from U into U .

Recall that positivity of Λ means that there exists $m > 0$ such that $\langle \Lambda u, u \rangle_U \geq m|u|_U^2$, $u \in \mathcal{D}(\Lambda)$ (see, for instance Section 3.3 in [31]). We also consider the Hilbert space $V = \mathcal{D}(\Lambda^{1/2}) = \text{Im}(\Lambda^{-1/2})$ endowed with the inner product

$$\langle h, k \rangle_V = \langle \Lambda^{1/2} h, \Lambda^{1/2} k \rangle_U, \quad h, k \in V$$

and its dual space V' which is again a Hilbert space. Note that $|\cdot|_{V'}$ is equivalent to $|\Lambda^{-1/2} \cdot|_U$. Indeed V' can be identified with the completion of U with respect to the norm $|\Lambda^{-1/2} \cdot|_U$ (see Section 3.4 in [31]). V' is also denoted by $\mathcal{D}(\Lambda^{-1/2})$. We have $V \subset H \simeq H' \subset V'$ with continuous inclusions; Λ can be extended to an unbounded self-adjoint operator on V' with domain V , which we still denote by Λ :

$$\Lambda : V \rightarrow V'. \quad (2.2)$$

We consider the linear stochastic wave equation in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $(\mathcal{F}_\tau)_{\tau \geq 0}$ satisfying the usual conditions. We have

$$\begin{cases} \frac{d^2 y}{d\tau^2}(\tau) = \Lambda y(\tau) + \dot{W}(\tau), \\ y(0) = x_0, \quad \frac{dy}{d\tau}(0) = x_1. \end{cases} \quad (2.3)$$

where $\{W(\tau) = W_\tau, \tau \geq 0\}$ is a cylindrical Wiener process in U with respect to the filtration $(\mathcal{F}_\tau)_{\tau \geq 0}$. The process W_t is formally given by “ $W_t = \sum_{j \geq 1} \beta_j(t) e_j$ ” where $\beta_j(t)$ are independent real Wiener processes and (e_j) denotes a basis in U (see [11] for more details). We need to introduce the Hilbert space H :

$$H = U \times V'$$

endowed with the inner product $\langle x, y \rangle_H = \langle x_1, y_1 \rangle_U + \langle x_2, y_2 \rangle_{V'}$ and norm $|x|_H = (\langle x, x \rangle_H)^{1/2}$, $x, y \in H$. In the sequel we will also denote $\langle \cdot, \cdot \rangle_H$ and $|\cdot|_H$ by $\langle \cdot, \cdot \rangle$ and $|\cdot|$.

According to [11], the equation (2.3) is well-posed in H thanks to Hypothesis 1. On the other hand, (2.3) is not well-posed in the more usual space $K = V \times U$ (i.e., solutions to (2.3) do not evolve in $K = V \times U$ even if $x_0 \in V$ and $x_1 \in U$). In H one considers the unbounded wave operator A which generates a unitary group e^{tA} :

$$\begin{aligned} \mathcal{D}(A) &= V \times U, \quad A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \text{for every } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A), \\ e^{tA} \begin{pmatrix} y \\ z \end{pmatrix} &= \begin{pmatrix} \cos \sqrt{\Lambda} t & \frac{1}{\sqrt{\Lambda}} \sin \sqrt{\Lambda} t \\ -\sqrt{\Lambda} \sin \sqrt{\Lambda} t & \cos \sqrt{\Lambda} t \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad t \in \mathbb{R}, \quad \begin{pmatrix} y \\ z \end{pmatrix} \in H. \end{aligned}$$

Let $G : U \rightarrow H$,

$$Gu = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u, \quad u \in U. \quad (2.4)$$

Equation (2.3) can be rewritten in an abstract form as

$$\begin{cases} dX_\tau = AX_\tau d\tau + GdW_\tau, & \tau \in [0, T], \\ X_0 = x \in H, \end{cases} \quad (2.5)$$

A solution to (2.5) is called Ornstein-Uhlenbeck process. We study (2.3) in H since the operators $Q_\tau = \int_0^\tau e^{sA} G G^* e^{sA^*} ds$, $\tau \geq 0$, are of trace class from H into H thanks to Hypothesis 1 (cf. Example 5.8 in [11]); here G^* denotes the adjoint operator of G . Thus the stochastic convolution

$$S_\tau = \int_0^\tau e^{(\tau-s)A} G dW_s \quad (2.6)$$

(i.e. the solution to (2.5) when $x = 0$) is well defined in H . Its law at time τ is the Gaussian measure $\mathcal{N}(0, Q_\tau)$ with mean 0 and covariance operator Q_τ (cf. [11]). Since

$$\sup_{t \in [0, T]} \|e^{tA} G\|_{L_2(U, H)} < \infty, \quad T > 0,$$

we can apply Theorem 5.11 in [11] and deduce that the process (S_τ) has a continuous version with values in H . Concerning the semilinear stochastic equation (1.4), we assume that

Hypothesis 2. $B : [0, T] \times H \rightarrow U$ is (Borel) measurable and bounded; moreover there exists $C > 0$ such that

$$|B(t, x + h) - B(t, x)|_U \leq C|h|_H^\alpha, \quad x, h \in H, \quad t \in [0, T],$$

for some $\alpha \in (2/3, 1)$. We also write that $B \in B_b([0, T]; C_b^\alpha(H, U))$ with $\alpha \in (2/3, 1)$.

Let $x \in H$. Recall that a (weak) mild solution to (1.4) is a tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$, where $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ is a stochastic basis on which it is defined a cylindrical U -valued \mathcal{F}_t -Wiener process W and a continuous \mathcal{F}_t -adapted H -valued process $X = (X_t) = (X_t)_{t \in [0, T]}$ such that, \mathbb{P} -a.s.,

$$X_t = e^{tA} x + \int_0^t e^{(t-s)A} G B(s, X_s) ds + \int_0^t e^{(t-s)A} G dW_s, \quad t \in [0, T]. \quad (2.7)$$

According to Chapter 1 in [28] (see also [24]) we say that equation (1.4) has a strong mild solution if, for every stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ on which there is defined an U -valued cylindrical \mathcal{F}_t -Wiener process W , there exists an H -valued continuous (\mathcal{F}_t) -adapted process $X = (X_t) = (X_t)_{t \in [0, T]}$ such that $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P}, W, X)$ is a weak mild solution. We also write $X_t^{0, x}$ or X_t^x instead of X_t . Similarly, we denote by $(X_\tau^{t, x})_{\tau \geq t}$ the solution to (1.4) starting from $x \in H$ at time $t \in [0, T]$.

Remark 2.1. Thanks to the boundedness of B we can apply the Girsanov Theorem as in [26]. For the infinite dimensional Girsanov theorem we refer to Proposition 7.1 in [28] and Section 10.3 in [11]. The Girsanov theorem allows to prove Theorem 5 in [28] which states that there always exists a weak mild solution, starting from any $x \in H$ (Theorem 5 in [28] even shows weak existence for random initial conditions). Moreover uniqueness in law holds for (1.4). To deduce such results by Theorem 5 of [28] we note the following facts: as f in [28] we can consider our $GB : [0, T] \times H \rightarrow H$; our space H can be the space $U = X = X_1$ used in [28]; the space U_0 in [28] can be our $\text{Im}G$; finally as cylindrical Wiener process of Theorem 5 in [28] we can consider our GW .

3 Examples

We present two classes of abstract semilinear stochastic wave equations that we can treat: the stochastic semilinear wave and plate equations. In Section 3.3 we also give a counterexample to uniqueness for deterministic semilinear wave equations with Hölder continuous coefficients.

3.1 Stochastic wave equations

We first deal with the semilinear stochastic wave equation as in Introduction, i.e.,

$$\begin{cases} \frac{\partial^2}{\partial \tau^2} y(\tau, \xi) = \frac{\partial^2}{\partial \xi^2} y(\tau, \xi) + b(\tau, \xi, y(\tau, \xi)) + \dot{W}(\tau, \xi), \\ y(\tau, 0) = y(\tau, 1) = 0, \\ y(0, \xi) = x_0(\xi), \\ \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), \quad \tau \in [0, T], \xi \in [0, 1]. \end{cases} \quad (3.1)$$

Comparing with (1.2), $\Lambda = -\frac{d^2}{dx^2}$ with Dirichlet boundary conditions, i.e., $\mathcal{D}(\Lambda) = H_0^1([0, 1]) \cap H^2([0, 1])$. Note that Λ^{-1} is of trace class since eigenvalues of Λ are $\lambda_n = n^2$, $n \geq 1$. Thus Hypothesis 1 holds.

We still denote by Λ its extension on $H^{-1}([0, 1])$ with domain

$$\mathcal{D}(\Lambda) = H_0^1([0, 1]), \quad \Lambda y = -\frac{\partial^2 y}{\partial \xi^2} \in H^{-1}([0, 1]), \quad \text{for every } y \in \mathcal{D}(\Lambda).$$

We consider $x_0 \in U = L^2([0, 1])$, $x_1 \in H^{-1}([0, 1])$. Writing $X_\tau(\xi) := \begin{pmatrix} y(\tau, \xi) \\ \frac{\partial}{\partial \tau} y(\tau, \xi) \end{pmatrix}$, according to Section 2, the reference Hilbert space for the solution is $H = L^2([0, 1]) \times H^{-1}([0, 1])$.

By considering $G : L^2([0, 1]) \rightarrow H$, $Gu = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u$ (cf. (2.4)) we can rewrite (3.1) in the abstract form (1.4) with $B(\tau, h) := b(\tau, \cdot, h_1(\cdot))$ and

$$GB(\tau, h)(\xi) := \begin{pmatrix} 0 \\ b(\tau, \xi, h_1(\xi)) \end{pmatrix}, \quad \xi \in [0, 1], \tau \in [0, T], h = (h_1, h_2) \in H. \quad (3.2)$$

It is easy to check that the next assumptions on b imply the validity of Hypothesis 2 for B .

Hypothesis 3.1. *The function $b : [0, T] \times [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and, for $\tau \in [0, T]$, a.e. $\xi \in [0, 1]$, the map $b(\tau, \xi, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. There exists c_1 bounded and measurable on $[0, 1]$, $\alpha \in (2/3, 1)$, such that, for $\tau \in [0, T]$ and a.e. $\xi \in [0, 1]$,*

$$|b(\tau, \xi, x) - b(\tau, \xi, y)| \leq c_1(\xi) |x - y|^\alpha,$$

$x, y \in \mathbb{R}$. Moreover $|b(\tau, \xi, x)| \leq c_2(\xi)$, for $\tau \in [0, T]$, $x \in \mathbb{R}$ and a.e. $\xi \in [0, 1]$, with $c_2 \in L^2([0, 1])$.

3.2 Stochastic plate equations

Let $D \subset \mathbb{R}^2$ be a bounded open domain with smooth boundary ∂D , which represents an elastic plate. We consider the following semilinear stochastic plate equation

$$\begin{cases} \frac{\partial^2 y}{\partial \tau^2}(\tau, \xi) = \Delta^2 y(\tau, \xi) + b(\tau, \xi, y(\tau, \xi)) + \dot{W}(\tau, \xi), \\ y(\tau, z) = 0, \quad \frac{\partial y}{\partial \nu}(\tau, z) = 0, \quad z \in \partial D, \\ y(0, \xi) = x_0(\xi), \quad \frac{\partial y}{\partial \tau}(0, \xi) = x_1(\xi), \quad \tau \in (0, T], \xi \in \overline{D}, \end{cases} \quad (3.3)$$

where Δ is the Laplacian in ξ , $\Delta^2 = \Delta(\Delta)$ is a fourth order operator, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on the boundary (we are considering the so-called clamped boundary conditions) and $\dot{W}(\tau, \xi)$ is a space-time white noise on $[0, T] \times D$. We remark that weak existence and uniqueness in law for non-linear stochastic plate equations with multiplicative noise have been established in [23].

Following Section III.8.4 in [2] we introduce $U = L^2(D)$ (the $L^2(D)$ space is defined with respect to the Lebesgue measure); the operator $\Lambda = \Delta^2$, with domain

$$\mathcal{D}(\Lambda) = H^4(D) \cap H_0^2(D)$$

is a positive self-adjoint operator ($H_0^2(D)$ is the closure of $C_0^\infty(D)$ in $H^2(D)$, see Definition 13.4.6 in [31]). One can prove that $\mathcal{D}(\Lambda^{1/2}) = H_0^2(D)$ (see page 172 in [2]). The topological dual of $H_0^2(D)$ will be indicated by $H^{-2}(D)$.

In order to check that Λ satisfies Hypothesis 1 we refer to [5]. Indeed a classical result by Courant (see page 460 of [5]) states that the eigenvalues λ_n of Λ have the asymptotic behaviour

$$\lambda_n \sim \frac{(4\pi n)^2}{f^2} \quad (3.4)$$

where f denotes the area of D (such behaviour depends on the size but not on the shape of the plate). It follows that Λ^{-1} is a trace class operator in $L^2(D)$. Proceeding as in Sections 2 and 3.1 we consider an extension of Λ to $H^{-2}(D)$ with domain $H_0^2(D)$.

The initial conditions of (3.3) are $x_0 \in L^2(D)$, $x_1 \in H^{-2}(D)$. The reference Hilbert space for the solution $X_\tau(\xi) := \begin{pmatrix} y(\tau, \xi) \\ \frac{\partial}{\partial \tau} y(\tau, \xi) \end{pmatrix}$ is $H = L^2(D) \times H^{-2}(D)$. By considering $G : L^2(D) \rightarrow H$, $Gu = \begin{pmatrix} 0 \\ u \end{pmatrix}$ (cf. (2.4)) we rewrite (3.3) in the abstract form (1.4) with $B(\tau, h) := b(\tau, \cdot, h_1(\cdot))$, $h = (h_1, h_2) \in H$. The assumptions we impose on b to verify Hypothesis 2 and get well-posedness for (3.3) are similar to Hypothesis 3.1.

Hypothesis 3.2. *The function $b : [0, T] \times D \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and, for $\tau \in [0, T]$ and a.e. $\xi \in D$, the map $b(\tau, \xi, \cdot) : \mathbb{R} \rightarrow \mathbb{R}$ is continuous. There exists c_1 bounded and measurable on D , $\alpha \in (2/3, 1)$, such that, for $\tau \in [0, T]$ and for a.e. $\xi \in D$,*

$$|b(\tau, \xi, x) - b(\tau, \xi, y)| \leq c_1(\xi) |x - y|^\alpha,$$

$x, y \in \mathbb{R}$. Moreover $|b(\tau, \xi, x)| \leq c_2(\xi)$, for $\tau \in [0, T]$ and a.e. $\xi \in D$, with $c_2 \in L^2(D)$.

3.3 A counterexample to well-posedness in the deterministic case

Let us consider the following semilinear deterministic wave equation for $\tau \in [0, T]$:

$$\begin{cases} \frac{\partial^2 y}{\partial \tau^2}(\tau, \xi) = \frac{\partial^2 y}{\partial \xi^2}(\tau, \xi) + b(\xi, y(\tau, \xi)) \\ y(\tau, 0) = y(\tau, \pi) = 0, \\ y(0, \xi) = 0, \quad \frac{\partial y}{\partial \tau}(0, \xi) = 0, \quad \xi \in [0, \pi]. \end{cases} \quad (3.5)$$

with

$$b(\xi, y) = 56 \sqrt[4]{\sin \xi} y^3 I_{\{|y| < 2T^8\}} + y I_{\{|y| < 2T^8\}} + 56 \sqrt[4]{8T^{24} \sin \xi} I_{\{|y| \geq 2T^8\}} + 2T^8 I_{\{|y| \geq 2T^8\}},$$

where $\xi \in [0, \pi]$, $y \in \mathbb{R}$; I_A is the indicator function of a set $A \subset \mathbb{R}$. Notice that b , which is independent of τ satisfies Hypothesis 3.1. It turns out that $y(\tau, \xi) \equiv 0$ and $y(\tau, \xi) = \tau^8 \sin \xi$ are both solutions to equation (3.5).

4 The H-valued transition semigroup for the stochastic wave equation

Here we prove some regularizing effects for the Ornstein-Uhlenbeck semigroup (R_t) related to equation (1.4) with $B = 0$ and acting on H -valued functions Φ .

In particular we show that the derivative of $R_t \Phi$ in the directions of U , i.e. $\nabla^G R_t \Phi$ takes values in the space of Hilbert-Schmidt operators $L_2(U, H)$ and provide a sharp estimate for $\|\nabla^G R_t \Phi\|_\infty$ when $t > 0$ (see in particular Lemma 4.2 and compare with Chapter 6 of [10] and Section 3 of [7]). From this result we deduce additional regularity results for second derivatives of $R_t \Phi$ (see Lemmas 4.3 and 4.4).

We first introduce the Ornstein-Uhlenbeck semigroup $R = (R_t)$ for H -valued functions:

$$R_\tau [\Phi](x) = R_\tau \Phi(x) = \mathbb{E} \Phi(X_\tau^{0,x}), \quad \Phi \in B_b(H, H), \quad x \in H, \quad \tau \geq 0, \quad (4.1)$$

where X , defined by (2.7), is the Ornstein-Uhlenbeck process (cf. [7]). Since X is time homogeneous, we have

$$R_{\tau-t} [\Phi](x) = \mathbb{E} \Phi(X_\tau^{t,x}), \quad \Phi \in B_b(H, H),$$

$\tau \geq t \geq 0$, $x \in H$. We now study the differentiability of $R_t[\Phi]$ for $t > 0$. To this aim we fix some notation.

If E is a Banach space and if $F : H \rightarrow E$ is Gâteaux differentiable at $x \in H$ we denote by $\nabla F(x) \in L(H, E)$ its Gâteaux derivative at x and by $\nabla_k F(x) = \nabla F(x)k$ its directional derivative along the direction $k \in H$:

$$\lim_{s \rightarrow 0} \frac{F(x + sk) - F(x)}{s} = \nabla_k F(x), \quad x \in H, \quad k \in H.$$

It is well-known that if $F : H \rightarrow E$ is Gâteaux differentiable on H and moreover the map: $x \mapsto \nabla F(x) \in L(H, E)$ is continuous from H into $L(H, E)$ then F is also Fréchet differentiable on H and the Gâteaux derivative coincides with the Fréchet derivative. Let $G : U \rightarrow H$, $Ga = \begin{pmatrix} 0 \\ a \end{pmatrix} \in H$, $a \in U$. If $F : H \rightarrow E$ is Gâteaux differentiable at $x \in H$ we set

$$\nabla_a^G F(x) = \nabla_{Ga} F(x) = \nabla F(x)Ga \in E, \quad a \in U, \quad x \in H. \quad (4.2)$$

Note that $\nabla^G F(x) = \nabla F(x)G \in L(U, E)$.

By the controllability of the abstract wave equation (see Appendix) we know that, for any $t > 0$,

$$e^{tA}(H) \subset Q_t^{1/2}(H). \quad (4.3)$$

Hence, see Chapter IV.2 in [34], $Q_t^{-1/2}e^{tA}$ is well defined, for any $t > 0$, and belongs to $L(H, H)$.

Let $\Phi \in B_b(H, H)$ and $x \in H$. Arguing as in Theorem 6.2.2 of [10], Section 9.4 in [11] and Section 3 of [7] one can prove the existence of the directional derivative of $R_t[\Phi]$:

$$\begin{aligned} & \lim_{s \rightarrow 0} \frac{R_t[\Phi](x + sk) - R_t[\Phi](x)}{s} \\ &= \nabla_k R_t[\Phi](x) = \nabla R_t[\Phi](x)k = \int_H \langle Q_t^{-\frac{1}{2}}e^{tA}k, Q_t^{-\frac{1}{2}}y \rangle \Phi(e^{tA}x + y) \mathcal{N}(0, Q_t)(dz), \quad k \in H, \quad t > 0. \end{aligned} \quad (4.4)$$

In the next result we will use (4.4) together with the following estimates (see Corollary A.2): for any $T > 0$ there exists $c > 0$ such that if $t \in (0, T]$, we have

$$|Q_t^{-1/2}e^{tA}h|_H \leq \frac{c}{t^{3/2}}|h|_H, \quad h \in H; \quad (4.5)$$

$$|Q_t^{-1/2}e^{tA}Ga|_H \leq \frac{c}{t^{1/2}}|Ga|_H = \frac{c}{t^{1/2}}|\Lambda^{-1/2}a|_U, \quad a \in U. \quad (4.6)$$

Lemma 4.1. Assume Hypothesis 1 and let $R = (R_t)$ be the OU semigroup defined in (4.1). If $\Phi \in B_b(H, H)$ and $t > 0$ then $R_t\Phi$ is Gâteaux differentiable on H and the Gâteaux derivative $\nabla R_t[\Phi](x) \in L(H, H)$ is given by (4.4). In particular $\nabla^G R_t[\Phi](x) \in L(U, H)$ is given by

$$\nabla_a^G R_t[\Phi](x) = \int_H \langle Q_t^{-\frac{1}{2}} e^{tA} G a, Q_t^{-\frac{1}{2}} y \rangle \Phi(e^{tA} x + y) \mathcal{N}(0, Q_t)(dz), \quad a \in U. \quad (4.7)$$

Moreover, for $t \in (0, T]$, we have

$$\sup_{x \in H} |\nabla_k R_t[\Phi](x)| \leq \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |k|_H, \quad k \in H; \quad (4.8)$$

$$\sup_{x \in H} |\nabla_a^G R_t[\Phi](x)| \leq \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |\Lambda^{-1/2} a|_U, \quad a \in U. \quad (4.9)$$

If in addition $\Phi \in C_b(H, H)$ then $R_t\Phi$ is Fréchet differentiable on H and $\nabla R_t[\Phi] \in C_b(H, L(H, H))$, $\nabla^G R_t[\Phi] \in C_b(H, L(U, H))$ for $t > 0$.

Proof. Let us fix $t \in (0, T]$ and $x \in H$. The integral in (4.4) defines a linear operator in $L(H, H)$. Let

$$I_{t,x}k := \int_H \langle Q_t^{-\frac{1}{2}} e^{tA} k, Q_t^{-\frac{1}{2}} y \rangle \Phi(e^{tA} x + y) \mathcal{N}(0, Q_t)(dy), \quad k \in H.$$

We have the well known estimate

$$\begin{aligned} |I_{t,x}k| &\leq \int_H |\langle Q_t^{-\frac{1}{2}} e^{tA} k, Q_t^{-\frac{1}{2}} y \rangle \Phi(e^{tA} x + y)| \mathcal{N}(0, Q_t)(dy) \\ &\leq \|\phi\|_\infty \left(\int_H |\langle Q_t^{-\frac{1}{2}} e^{tA} k, Q_t^{-\frac{1}{2}} y \rangle|^2 \mathcal{N}(0, Q_t)(dy) \right)^{1/2} = \|\phi\|_\infty |Q_t^{-\frac{1}{2}} e^{tA} k|_H \leq \frac{c}{t^{\frac{3}{2}}} \|\Phi\|_\infty |k|_H. \end{aligned} \quad (4.10)$$

Computing the directional derivative as in (4.4) we obtain the Gâteaux differentiability of $R_t[\Phi]$ at x and estimates (4.8) and (4.9) follow. If $\Phi \in C_b(H, H)$ we compute, for any $k \in H$, $|k| = 1$, $z \in H$,

$$\begin{aligned} &|I_{t,x}k - I_{t,x+z}k|^2 \\ &\leq \left| \int_H \langle Q_t^{-\frac{1}{2}} e^{tA} k, Q_t^{-\frac{1}{2}} y \rangle [\Phi(e^{tA} x + y) - \Phi(e^{tA}(x+z) + y)] \mathcal{N}(0, Q_t)(dy) \right|^2 \\ &\leq \int_H |\langle Q_t^{-\frac{1}{2}} e^{tA} k, Q_t^{-\frac{1}{2}} y \rangle|^2 \mathcal{N}(0, Q_t)(dy) \int_H |\Phi(e^{tA} x + y) - \Phi(e^{tA} x + e^{tA} z + y)|^2 \mathcal{N}(0, Q_t)(dy) \\ &\leq \frac{c^2 |k|_H^2}{t^3} \int_H \sup_{x \in H} |\Phi(e^{tA} x + y) - \Phi(e^{tA} x + e^{tA} z + y)|^2 \mathcal{N}(0, Q_t)(dy) \end{aligned}$$

and so by the dominated convergence theorem we obtain easily

$$\lim_{z \rightarrow 0} \sup_{y \in H} \sup_{|k|_H=1} |I_{t,y}k - I_{t,y+z}k| = 0. \quad (4.11)$$

By a well known result the Gâteaux derivative $I_{t,x}$ is indeed a Fréchet derivative. Moreover, by (4.11) $\nabla R_t[\Phi] \in C_b(H, L(H, H))$. \square

Next we improve the regularity of $\nabla^G R_t[\Phi]$. To simplify notation we set

$$\Gamma_t = Q_t^{-\frac{1}{2}} e^{tA}, \quad \mu_t = \mathcal{N}(0, Q_t).$$

Lemma 4.2. Under the assumptions of Lemma 4.1 if $\Phi \in B_b(H, H)$ we have $\nabla^G R_t[\Phi](x) \in L_2(U, H)$, $x \in H$, $t > 0$, and

$$\sup_{x \in H} \|\nabla^G R_t[\Phi](x)\|_{L_2(U, H)} \leq \frac{c}{t^{\frac{1}{2}}} \|\Phi\|_\infty, \quad t > 0. \quad (4.12)$$

Moreover, if in addition $\Phi \in C_b(H, H)$ then $\nabla^G R_t[\Phi] \in C_b(H, L_2(U, H))$.

Proof. Let (e_k) be a basis in U . We have, using also (4.6),

$$\begin{aligned} \sum_{k \geq 1} |\nabla_{e_k}^G R_t[\Phi](x)|^2 &\leq \sum_{k \geq 1} \int_H |\langle Q_t^{-\frac{1}{2}} e^{tA} G e_k, Q_t^{-\frac{1}{2}} y \rangle|^2 |\Phi(e^{tA} x + y)|^2 \mu_t(dz) \\ &\leq \frac{c}{t} \|\Phi\|_\infty^2 \sum_{k \geq 1} |G e_k|_H^2 = \frac{c}{t} \|\Phi\|_\infty^2 \sum_{k \geq 1} |\Lambda^{-1/2} e_k|_U^2 = \frac{c}{t} \|\Phi\|_\infty^2 \|\Lambda^{-1/2}\|_{L_2(U, U)}^2 \end{aligned}$$

and this shows the first assertion and estimate (4.12). If $\Phi \in C_b(H, H)$ to prove the uniform continuity of $\nabla^G R_t[\Phi]$ we argue as for (4.11); by using also the dominated convergence theorem, we obtain

$$\lim_{z \rightarrow 0} \sup_{x \in H} \sum_{k \geq 1} |\nabla_{e_k}^G R_t[\Phi](x+z) - \nabla_{e_k}^G R_t[\Phi](x)|^2 = \lim_{z \rightarrow 0} \sup_{x \in H} \|\nabla^G R_t[\Phi](x+z) - \nabla^G R_t[\Phi](x)\|_{L_2(U, H)}^2 = 0$$

and we conclude that $\nabla^G R_t[\Phi] \in C_b(H, L_2(U, H))$. \square

In a similar way we get

Lemma 4.3. *Under the assumptions of Lemma 4.1 let $t > 0$. If $\Phi \in C_b(H, H)$ and $\xi \in U$ the mapping:*

$$x \mapsto \nabla_\xi^G R_t[\Phi](x)$$

with values in H is Fréchet differentiable on H . The second order directional derivatives are

$$\nabla_k \nabla_\xi^G R_t[\Phi](x) = \int_H (\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G \xi, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G \xi \rangle) \Phi(e^{tA} x + y) \mu_t(dy), \quad (4.13)$$

for $x, k \in H$, $\xi \in U$. Moreover, for each $x, k \in H$, the map: $\xi \rightarrow \nabla_k \nabla_\xi^G R_t[\Phi](x)$ belongs to $L_2(U, H)$ and

$$\sup_{x \in H} \|\nabla_k \nabla_\xi^G R_t[\Phi](x)\|_{L_2(U, H)} \leq \frac{c|k|_H}{t^2} \|\Phi\|_\infty, \quad k \in H, \quad (4.14)$$

$$\begin{aligned} &\lim_{x \rightarrow 0} \sup_{y \in H} \|\nabla \nabla_\xi^G R_t[\Phi](x+y) - \nabla \nabla_\xi^G R_t[\Phi](y)\|_{L(H, H)} \\ &= \lim_{x \rightarrow 0} \sup_{y \in H} \sup_{|k|=1} |\nabla_k \nabla_\xi^G R_t[\Phi](x+y) - \nabla_k \nabla_\xi^G R_t[\Phi](y)|_H = 0, \quad \xi \in U. \end{aligned} \quad (4.15)$$

Proof. Let (e_j) be a basis in U and fix $t > 0$, $x \in H$. First define $J_{t,x,k,\xi}$ as the integral in the right hand side of (4.13). Proceeding as in the proof of Lemma 4.2 it is not difficult to show that

$$k \mapsto J_{t,x,k,\xi}, \quad (4.16)$$

is linear from H into $L_2(U, H)$. Moreover, using (4.5) (4.6) and the Hölder inequality, we get

$$\begin{aligned} \sum_{j \geq 1} |J_{t,x,k,e_j}|^2 &\leq \sum_{j \geq 1} \int_H |\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G e_j, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G e_j \rangle|^2 |\Phi(e^{tA} x + y)|^2 \mu_t(dy) \\ &\leq \frac{c|k|^2}{t^4} \|\Phi\|_\infty^2 \sum_{j \geq 1} |\Lambda^{-1/2} e_j|_U^2 \leq \frac{c|k|^2}{t^4} \|\Phi\|_\infty^2 \|\Lambda^{-1/2}\|_{L_2(U, U)}^2. \end{aligned} \quad (4.17)$$

Thus the linear operator in (4.16) is a bounded operator from H into $L_2(U, H)$.

Let now $\xi \in U$. Arguing as in Section 9.4 of [11] and Section 3 in [7] we find that

$$\lim_{s \rightarrow 0} \frac{\nabla_\xi^G R_t[\Phi](x+sk) - \nabla_\xi^G R_t[\Phi](x)}{s} = J_{t,x,k,\xi}, \quad k \in H.$$

Moreover, for any $z \in H$, $\xi \in U$,

$$\begin{aligned} &|J_{t,x,k,\xi} - J_{t,x+z,k,\xi}|^2 \\ &= \left| \int_H (\langle \Gamma_t k, Q_t^{-\frac{1}{2}} y \rangle \langle \Gamma_t G \xi, Q_t^{-\frac{1}{2}} y \rangle - \langle \Gamma_t k, \Gamma_t G \xi \rangle) [\Phi(e^{tA} x + e^{tA} z + y) - \Phi(e^{tA} x + y)] \mu_t(dy) \right|^2 \\ &\leq \frac{c|k|_H^2}{t^4} |\Lambda^{-1/2} \xi|_U^2 \int_H |\Phi(e^{tA} x + y) - \Phi(e^{tA} x + e^{tA} z + y)|^2 \mu_t(dy) \end{aligned}$$

and so

$$\lim_{z \rightarrow 0} \sup_{x \in H} \sup_{|k|=1} |J_{t,x,k,\xi} - J_{t,x+z,k,\xi}|^2 = 0. \quad (4.18)$$

This shows in particular that the mapping $x \mapsto \nabla_\xi^G R_t[\Phi](x)$ with values in H is Fréchet differentiable on H and that (4.15) holds. Moreover, (4.14) follows from (4.17). \square

Using interpolation theory (see also (2.1)) we can improve the previous estimates in the case when Φ is Hölder continuous.

Lemma 4.4. *Under the assumptions of Lemma 4.1 let $\Phi \in C_b^\alpha(H, H)$, $\alpha \in (0, 1)$. We have the following estimates, for $t \in (0, T]$,*

$$\begin{aligned} \sup_{x \in H} |\nabla_k R_t[\Phi](x)|_H &\leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi\|_\alpha |k|_H, \quad k \in H; \\ \sup_{x \in H} \|\nabla_k \nabla_\xi^G R_t[\Phi](x)\|_{L_2(U, H)} &\leq \frac{c}{t^{\frac{4-3\alpha}{2}}} \|\Phi\|_\alpha |k|_H, \quad k \in H. \end{aligned} \quad (4.19)$$

Proof. Let us fix $t \in (0, T]$, $k \in H$ and $\xi \in U$. Using the OU process X defined by (2.7) we can define the Ornstein-Uhlenbeck semigroup (P_t) acting on scalar functions $\phi \in B_b(H)$:

$$P_\tau[\phi](x) = P_\tau \phi(x) = \mathbb{E} \phi(X_\tau^{0,x}), \quad \phi \in B_b(H), \quad \tau \geq 0.$$

For $h \in H$, we introduce the scalar function $\Phi_h(x) = \langle \Phi(x), h \rangle$, $x \in H$, which belongs to $C_b^\alpha(H)$ with $\|\Phi_h\|_\alpha \leq \|\Phi\|_\alpha |h|$. We note as in Section 3 of [7] that

$$\langle \nabla_k R_t[\Phi](x), h \rangle = \nabla_k P_t[\Phi_h](x), \quad x \in H.$$

Let $k \in H$. To prove the first estimate we consider the linear operators

$$\nabla_k P_t : C_b^1(H) \rightarrow C_b(H), \quad \nabla_k P_t : C_b(H) \rightarrow C_b(H)$$

When $\phi \in C_b^1(H)$, we find that

$$\nabla_k P_t[\phi](x) = \lim_{s \rightarrow 0} \int_H \frac{\phi(e^{tA}x + se^{tA}k + y) - \phi(e^{tA}x + y)}{s} \mu_t(dy) = \int_H \langle \nabla \phi(e^{tA}x + y), e^{tA}k \rangle \mu_t(dy)$$

and we get the estimate

$$\sup_{x \in H} |\nabla_k P_t[\phi](x)| \leq C \|\nabla \phi\|_\infty |k|_H, \quad \phi \in C_b^1(H). \quad (4.20)$$

Interpolating between (4.20) and

$$\sup_{x \in H} |\nabla_k P_t[f](x)| \leq \frac{c}{t^{\frac{3}{2}}} \|f\|_\infty |k|_H, \quad f \in C_b(H),$$

we obtain thanks to (2.1)

$$\sup_{x \in H} |\nabla_k P_t[\psi](x)| \leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\psi\|_\alpha |k|_H, \quad \psi \in C_b^\alpha(H).$$

If we consider now $\psi = \Phi_h$, we have, for each $x \in H$, $h \in H$,

$$|\langle \nabla_k R_t[\Phi](x), h \rangle| = |\nabla_k P_t[\Phi_h](x)| \leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi_h\|_\alpha |k|_H \leq \frac{c}{t^{\frac{3}{2}(1-\alpha)}} \|\Phi\|_\alpha |k|_H |h|_H.$$

By taking the supremum over $\{h \in H : |h|_H = 1\}$ we get the first estimate in (4.19).

To prove the second estimate we fix $k \in H$, $\xi \in U$ and introduce the linear operators

$$\nabla_k \nabla_\xi^G P_t : C_b^1(H) \rightarrow C_b(H), \quad \nabla_k \nabla_\xi^G P_t : C_b(H) \rightarrow C_b(H).$$

When $\phi \in C_b^1(H)$ we know that

$$\nabla_k \nabla_\xi^G P_t[\phi](x) = \int_H \langle \Gamma_t G \xi, Q_t^{-\frac{1}{2}} y \rangle \nabla_{e^{tA}k} \phi(e^{tA}x + y) \mu_t(dy).$$

and so

$$\sup_{x \in H} |\nabla_k(\nabla_\xi^G P_t[\phi])(x)| \leq \frac{c}{t^{1/2}} \|\phi\|_{C_b^1} |k|_H |G\xi|_H = \frac{c}{t^{1/2}} \|\phi\|_{C_b^1} |k|_H |\Lambda^{-1/2}\xi|_U \quad (4.21)$$

(see (4.6)). Interpolating between (4.21) and

$$\sup_{x \in H} |\nabla_k(\nabla_\xi^G P_t[\phi])(x)| \leq \frac{c|k|_H |\Lambda^{-1/2}\xi|_U}{t^2} \|\phi\|_\infty$$

(cf. (4.17)) we obtain

$$\sup_{x \in H} |\nabla_k(\nabla_\xi^G P_t[\psi])(x)| \leq \frac{c}{t^{\frac{4-3\alpha}{2}}} \|\psi\|_\alpha |k|_H |\Lambda^{-1/2}\xi|_U, \quad \psi \in C_b^\alpha(H), \quad (4.22)$$

since $\frac{1}{2}\alpha + 2(1-\alpha) = 2 - \frac{3}{2}\alpha$. Now for $x \in H$, we compute using a basis (e_j) in U

$$\begin{aligned} \|\nabla_k(\nabla^G R_t[\Phi])(x)\|_{L_2(U,H)}^2 &= \sum_{j \geq 1} |\nabla_k(\nabla_{e_j}^G R_t[\Phi])(x)|_H^2 \\ &= \sum_{j \geq 1} \sup_{|h|_H=1} |\langle \nabla_k(\nabla_{e_j}^G R_t[\Phi])(x), h \rangle|^2 = \sum_{j \geq 1} \sup_{|h|_H=1} |\nabla_k(\nabla_{e_j}^G P_t[\Phi_h])(x)|^2 \\ &\leq \frac{c|k|_H^2}{t^{4-3\alpha}} \sum_{j \geq 1} \sup_{|h|_H=1} (\|\Phi_h\|_\alpha^2 |\Lambda^{-1/2}e_j|_U^2) \\ &\leq \frac{c|k|_H^2}{t^{4-3\alpha}} \|\Phi\|_\alpha^2 \sum_{j \geq 1} |\Lambda^{-1/2}e_j|_U^2 = \frac{c|k|_H^2}{t^{4-3\alpha}} \|\Phi\|_\alpha^2 \|\Lambda^{-1/2}\|_{L_2(U,U)}^2. \end{aligned}$$

The second estimate in (4.19) follows easily. \square

We consider the following integral equation which will be important in Section 5.1:

$$u(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)A} G B(s, \cdot) \right] (x) ds + \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G u(s, \cdot) B(s, \cdot) \right] (x) ds, \quad (4.23)$$

where $\nabla^G u(s, x) B(s, x) = \nabla_{B(s,x)}^G u(s, x)$, $(s, x) \in [0, T] \times H$ (see (4.2)).

Using the previous lemmas, we will solve the equation in the Banach space E_0 consisting of all $u \in B_b([0, T] \times H, H)$ such that $u(t, \cdot)$ is Fréchet differentiable on H , with Fréchet derivative $\nabla u \in B_b([0, T] \times H, L(H, H))$ and $\nabla^G u \in B_b([0, T] \times H, L_2(U, H))$. Moreover, for each $\xi \in U$, $t \in [0, T]$, the mapping:

$$x \mapsto \nabla_\xi^G u(t, x) \text{ is Fréchet differentiable on } H \text{ with } \sup_{(t,x) \in [0,T] \times H} \sup_{|\xi|_U=1} \|\nabla \nabla_\xi^G u(t, x)\|_{L(H,H)} < \infty. \quad (4.24)$$

Let $\beta \geq 0$ to be fixed later. It is not difficult to prove that E_0 is a Banach space endowed with the norm

$$\begin{aligned} \|u\|_{E_0, \beta} &= \sup_{(t,x) \in [0,T] \times H} e^{\beta t} |u(t, x)| + \sup_{(t,x) \in [0,T] \times H} e^{\beta t} \|\nabla u(t, x)\|_{L(H,H)} \\ &+ \sup_{(t,x) \in [0,T] \times H} e^{\beta t} \|\nabla^G u(t, x)\|_{L_2(U,H)} + \sup_{(t,x) \in [0,T] \times H} \sup_{|\xi|_U=1} e^{\beta t} \|\nabla \nabla_\xi^G u(t, x)\|_{L(H,H)}. \end{aligned} \quad (4.25)$$

Theorem 4.5. *Let Hypotheses 1 and 2 hold true. There exists a unique solution $u \in E_0$ to (4.23). Moreover, for each $x, k \in H$, $t \in [0, T]$, the map: $\xi \rightarrow \nabla_k \nabla_\xi^G u(t, x)$ belongs to $L_2(U, H)$ and, for any $k \in H$, the mapping:*

$$(t, x) \mapsto \nabla_k \nabla^G u(t, x) \text{ is measurable from } [0, T] \times H \text{ into } L_2(U, H) \quad (4.26)$$

and

$$\sup_{x \in H, t \in [0, T]} \|\nabla_k \nabla^G u(t, x)\|_{L_2(U, H)} \leq c|k|, \quad k \in H,$$

for some $c > 0$ (independent of k).

Finally, there exists a function $h(r) = h(r, \alpha) > 0$, $r \geq 0$, such that $h(r) \rightarrow 0$ as $r \rightarrow 0^+$ and if $S \in [0, T]$ verifies $h(T - S) \cdot (\sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha) \leq 1/4$, then

$$\sup_{t \in [S, T], x \in H} \|\nabla v(t, x)\|_{L(H, H)} \leq 1/3. \quad (4.27)$$

Remark 4.6. Actually it would be possible to prove more regularity for the solution to equation (4.23) like the joint continuity in (t, x) of u and of its derivatives. However the proof would become more involved. On the other hand, the regularity of u as stated in Theorem 4.5 is enough to prove our pathwise result on (1.4) (see Section 5.1 and Theorem 6.3).

Proof. We introduce the following operator \mathcal{T} defined on E_0 :

$$\mathcal{T}u(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)A} G B(s, \cdot) \right] (x) ds + \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G u(s, \cdot) B(s, \cdot) \right] (x) ds,$$

$u \in E_0$, $(t, x) \in [0, T] \times H$. Note that in particular

$$\nabla^G u(s, \cdot) B(s, \cdot) \text{ is } \alpha\text{-H\"older continuous and bounded from } H \text{ into } H, \quad (4.28)$$

uniformly with respect to $s \in [0, T]$, where α is given in Hypothesis 2.

By using the Lemmas 4.3 and 4.4 and the fact that $\alpha > 2/3$, it is not difficult to prove that $\mathcal{T} : E_0 \rightarrow E_0$. Let us check that for a suitable value of β the map \mathcal{T} is a strict contraction (see (4.25)). We have to consider $\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{E_{0,\beta}}$, $u_1, u_2 \in E_0$; we only treat the term

$$\sup_{t, x} \sup_{|\xi|_U=1} e^{\beta t} \|\nabla \nabla_\xi^G [\mathcal{T}u_1 - \mathcal{T}u_2](t, x)\|_{L(H, H)}.$$

Indeed the other terms of $\|\mathcal{T}u_1 - \mathcal{T}u_2\|_{E_{0,\beta}}$ can be estimated in a similar way. We have

$$\begin{aligned} & e^{\beta t} \|\nabla \nabla_\xi^G [\mathcal{T}u_1(t, x) - \mathcal{T}u_2(t, x)]\|_{L(H, H)} \\ & \leq \int_t^T e^{-\beta(s-t)} \left\| \nabla \nabla_\xi^G R_{s-t} \left[e^{-(s-t)A} e^{\beta s} \{ \nabla^G u_1(s, \cdot) - \nabla^G u_2(s, \cdot) \} B(s, \cdot) \right] (x) \right\|_{L(H, H)} ds \\ & \leq \int_t^T \frac{c e^{-\beta(s-t)}}{(s-t)^{\frac{4-3\alpha}{2}}} ds \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha \|u_1 - u_2\|_{E_{0,\beta}} \\ & \leq C_{\beta, T} \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha \|u_1 - u_2\|_{E_{0,\beta}}, \end{aligned}$$

where $C_{\beta, T} > 0$ tends to 0 as $\beta \rightarrow +\infty$. Choosing β large enough, we can apply the fixed point theorem and obtain that there exists a unique solution $u \in E_0$.

In order to prove (4.27), we first introduce $\|u\|_{E_{0,0,S,T}}$ which is defined as $\|u\|_{E_{0,0}}$ in (4.25) (with $\beta = 0$) but taking all the supremums over $[S, T] \times H$ instead of $[0, T] \times H$. We proceed as before:

$$\begin{aligned} \|u\|_{E_{0,0,S,T}} & \leq \sup_{t \in [S, T]} \int_t^T \frac{c}{(s-t)^{\frac{4-3\alpha}{2}}} ds \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha (\|u_0\|_{E_{0,0,S,T}} + 1) \\ & \leq h(T - S) \sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha (\|u_0\|_{E_{0,0,S,T}} + 1), \end{aligned}$$

where $h(r) = \int_0^r \frac{c}{s^{\frac{4-3\alpha}{2}}} ds$; now (4.27) follows since we have $\frac{3}{4}\|u\|_{E_{0,0,S,T}} \leq 1/4$. \square

5 The related infinite dimensional forward-backward system

In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let us consider the following forward-backward system (FBSDE) with forward and backward equations both taking values in H ,

$$\begin{cases} d\Xi_\tau^{t,x} = A\Xi_\tau^{t,x}d\tau + GdW_\tau, & \tau \in [t, T], \\ \Xi_t^{t,x} = x, \\ -dY_\tau^{t,x} = -AY_\tau^{t,x} + GB(\tau, \Xi_\tau^{t,x})d\tau + Z_\tau^{t,x}B(\tau, \Xi_\tau^{t,x})d\tau - Z_\tau^{t,x}dW_\tau, & \tau \in [0, T], \\ Y_T^{t,x} = 0, \end{cases} \quad (5.1)$$

where $t \in [0, T]$, $x \in H$, and the forward equation is the abstract formulation of the wave equation (2.3) given in (2.5) under Hypothesis 1; here $B : [0, T] \times H \rightarrow U$ is (Borel) measurable and satisfies

$$B(t, \cdot) \in C_b(H, U), \quad t \in [0, T], \quad \|B\|_\infty = \sup_{t \in [0, T] \times H} |B(t, x)|_U < \infty \quad (5.2)$$

(clearly, Hypothesis 2 implies (5.2)); G is defined by (2.4) and W is a cylindrical Wiener process in U . We extend $\Xi_\tau^{t,x}$ to the whole $[0, T]$ by setting $\Xi_\tau^{t,x} = x$ for $0 \leq \tau \leq t$, in order to have $(Y^{t,x}, Z^{t,x})$ well defined on $[0, T]$. The precise meaning of the BSDE in (5.1) is given by its mild formulation: for $\tau \in [0, T]$

$$Y_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A} GB(s, \Xi_s^{t,x}) ds + \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x} B(s, \Xi_s^{t,x}) ds - \int_\tau^T e^{-(s-\tau)A} Z_s^{t,x} dW_s, \quad (5.3)$$

\mathbb{P} -a.s. (cf. [22], [17], [18], [20] and the references therein). The solution of (5.3) will be a pair of processes $(Y^{t,x}, Z^{t,x})$ (see Proposition 5.1). Notice that in order to give sense to the BSDE in (5.1) as it is done in (5.3), we need that A is the generator of a C_0 -group of bounded linear operators, so that $-A$ is the generator of a C_0 -semigroup of bounded linear operators.

We also refer to this BSDE as BSDE in a Markovian framework, since the pair of processes $(Y^{t,x}, Z^{t,x})$ depends on the Markov process $\Xi^{t,x}$. We endow $(\Omega, \mathcal{F}, \mathbb{P})$ with the natural filtration (\mathcal{F}_t^W) of W , augmented in the usual way with the family of \mathbb{P} -null sets of \mathcal{F} . All the concepts of measurability, e.g. predictability, are referred to this filtration. We denote by $L_{\mathcal{P}}^2(\Omega, C([0, T], H))$ the space of all predictable H -valued processes Y with continuous paths and such that

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} |Y_\tau|^2 \right] = \|Y\|_{L_{\mathcal{P}}^2(\Omega, C([0, T], H))}^2 < \infty.$$

The space $L_{\mathcal{P}}^2(\Omega, C([0, T], H))$ is a Banach space endowed with the norm $\|\cdot\|_{L_{\mathcal{P}}^2(\Omega, C([0, T], H))}$. On the other hand, $L_{\mathcal{P}}^2(\Omega \times [0, T], L_2(U, H))$ is the usual L^2 -space of all predictable processes Z with values in $L_2(U, H)$. We also define the space $\mathcal{G}^{0,1}([0, T] \times H, H)$, see [17] Section 2.2, as the subspace of $C([0, T] \times H, H)$ consisting of all functions f which are Gâteaux differentiable with respect to x and such that the map $\nabla_x f : [0, T] \times H \rightarrow L(H, H)$ is strongly continuous. Similarly one can define the space $\mathcal{G}^{0,1}([0, T] \times H, \mathbb{R}) \subset C([0, T] \times H, \mathbb{R})$.

Following [22], it is immediate to get existence and pathwise uniqueness of a solution $(Y^{t,x}, Z^{t,x})$ to the Markovian BSDE (5.3). Moreover we can show regular dependence on the initial datum x of the solution to the forward equation in (5.1).

Proposition 5.1. *Assume Hypothesis 1 and let B be as in (5.2). Let $t \in [0, T]$ and $x \in H$. Consider the H -valued BSDE (5.3). Then there exists a unique solution $(Y^{t,x}, Z^{t,x}) \in L_{\mathcal{P}}^2(\Omega, C([0, T], H)) \times L_{\mathcal{P}}^2(\Omega \times [0, T], L_2(U, H))$. Moreover the following estimates hold true:*

$$\mathbb{E} \left[\sup_{\tau \in [0, T]} |Y_\tau^{t,x}|^2 \right] + \mathbb{E} \int_0^T \|Z_\tau^{t,x}\|_{L_2(U, H)}^2 d\tau \leq C_T \|B\|_\infty. \quad (5.4)$$

In addition, the map: $(t, x) \mapsto Y_t^{t,x}$, $[0, T] \times H \rightarrow H$, is deterministic. If we further assume that

$$\text{the map: } x \mapsto B(\tau, x), \quad H \rightarrow U, \text{ is Gâteaux differentiable on } H, \text{ for all } \tau \in [0, T], \quad (5.5)$$

then, for any $t \in [0, T]$, the map

$$x \mapsto (Y^{t,x}, Z^{t,x}), \quad H \rightarrow L_{\mathcal{P}}^2(\Omega, C([0, T], H)) \times L_{\mathcal{P}}^2(\Omega \times [0, T], L_2(U, H)) \quad (5.6)$$

is Gâteaux differentiable on H . Moreover, assuming (5.5), the map: $(t, x) \mapsto Y_t^{t,x}$ belongs to $\mathcal{G}^{0,1}([0, T] \times H, H)$.

Proof. Existence and uniqueness of a solution come directly from Lemma 2.1 and Proposition 2.1 in [22], that we can apply since B is bounded. Estimate (5.4) follows also from [20], Proposition 3.5 and estimate (3.21). Since the process $\Xi^{t,x}$ is $\mathcal{F}_{t,T}^W$ -measurable (where $\mathcal{F}_{t,T}^W$ is the σ -algebra generated by $W_r - W_t$, $r \in [t, T]$, augmented with the \mathbb{P} -null sets), it turns out that $Y_t^{t,x}$ is measurable both with respect to $\mathcal{F}_{t,T}^W$ and \mathcal{F}_t^W ; it follows that $Y_t^{t,x}$ is indeed deterministic.

When B is also differentiable with respect to x (see (5.5)) then the differentiability properties follow by [17], Propositions 4.8 and 5.2, which can be applied in the same way also when the BSDE is H -valued. \square

Let $(Y^{t,x}, Z^{t,x})$ be the solution of (5.1) assuming only Hypothesis 1 and (5.2). By the previous result we can define the deterministic function $v : [0, T] \times H \rightarrow H$,

$$v(t, x) = Y_t^{t,x} \in H, \quad (t, x) \in [0, T] \times H. \quad (5.7)$$

Assuming also the differentiability condition (5.5), the map defined in (5.6) is in particular continuous and it is standard to check the following useful identities: for any $0 \leq t \leq s \leq \tau \leq T$,

$$Y_\tau^{t,x} = Y_\tau^{s, \Xi_s^{t,x}}, \quad Z_\tau^{t,x} = Z_\tau^{s, \Xi_s^{t,x}}, \quad \mathbb{P} - \text{a.s.} \quad (5.8)$$

The proof of (5.8) can be performed as for the real valued BSDEs (see [17], formula (5.3)), and it is related to the fact that the value of the processes $Y^{t,x}$ and $Z^{t,x}$ on the time interval $[s, T]$ is uniquely determined by the values of $\Xi^{t,x}$ on the same interval.

Moreover, if we assume differentiability of B (see (5.5)), we get in particular that, for $t \in [0, T]$, $v(t, \cdot) : H \rightarrow H$ is Gâteaux differentiable on H , and, moreover, applying (5.8), we have:

$$v(\tau, \Xi_\tau^{t,x}) = Y_\tau^{t,x}, \quad \tau \in [t, T], \quad \mathbb{P} - \text{a.s.} \quad (5.9)$$

Now we want to prove that the derivative $\nabla^G v(\tau, \Xi_\tau^{t,x})$ can be identified with $Z_\tau^{t,x}$ (see (4.2)). At first we prove such identification assuming that B is also differentiable (see (5.5)). Then in Theorem 5.4 we show that such identification holds true only assuming (5.2).

Remark 5.2. The next identification property with B differentiable (see (5.5)) is here presented for the markovian BSDE (5.3) related to the Ornstein Uhlenbeck wave process; it remains true for general linear Markovian BSDEs with differentiable coefficients and final datum which is related to a forward stochastic equation with additive noise, A generator of a strongly continuous semigroup, A instead of $-A$ in the backward equation, and Lipschitz continuous and Gâteaux differentiable drift.

Lemma 5.3. *Let v be defined in (5.7) and assume that all the hypotheses of Proposition 5.1, including the differentiability of B (see (5.5)) hold true. Let $(Y^{t,x}, Z^{t,x})$ be the solution of (5.3). Then, for any $\tau \in [0, T]$, a.e., we have, \mathbb{P} -a.s.,*

$$\nabla^G v(\tau, \Xi_\tau^{t,x}) = Z_\tau^{t,x} \text{ in } L_2(U, H). \quad (5.10)$$

Proof. The result can be seen as an extension of Theorem 6.1 in [18] (see also Theorem 6.2 in [17]) to the case of an H -valued BSDE. Let $\xi \in U$ and consider the real Wiener process $(W_\tau^\xi)_{\tau \geq 0}$, where

$$W_\tau^\xi := \langle \xi, W_\tau \rangle_U.$$

Let $h \in H$. Using the group property if we set $\tilde{Y}_\tau^{t,x} = e^{-\tau A} Y_\tau^{t,x}$ we have, for $\tau \in [0, T]$, \mathbb{P} -a.s.,

$$\begin{aligned} \tilde{Y}_\tau^{t,x} &= \int_\tau^T e^{-sA} GB(s, \Xi_s^{t,x}) ds + \int_\tau^T e^{-sA} Z_s^{t,x} B(s, \Xi_s^{t,x}) ds - \int_\tau^T e^{-sA} Z_s^{t,x} dW_s \\ &= \tilde{Y}_0^{t,x} - \int_0^\tau e^{-sA} GB(s, \Xi_s^{t,x}) ds - \int_0^\tau e^{-sA} Z_s^{t,x} B(s, \Xi_s^{t,x}) ds + \int_0^\tau e^{-sA} Z_s^{t,x} dW_s \end{aligned}$$

and so

$$\begin{aligned} \langle \tilde{Y}_\tau^{t,x}, h \rangle &= \langle \tilde{Y}_0^{t,x}, h \rangle - \int_0^\tau \langle e^{-sA} GB(s, \Xi_s^{t,x}), h \rangle ds - \int_0^\tau \langle e^{-sA} Z_s^{t,x} B(s, \Xi_s^{t,x}), h \rangle ds \\ &\quad + \int_0^\tau \langle dW_s, (Z_s^{t,x})^* e^{-sA^*} h \rangle_U ds \end{aligned}$$

(A^* denotes the adjoint of A , $A^* = -A$). We study the joint quadratic variation between $\tilde{Y}^{t,x}$ and W^ξ (see, for instance, page 638 in [18]). We find, arguing as in the proof of Proposition 17 of [9],

$$\langle \tilde{Y}^{t,x}, W^\xi \rangle_\tau = \int_0^\tau \langle \xi, (Z_s^{t,x})^* e^{-sA^*} h \rangle_U ds = \int_0^\tau \langle e^{-sA} Z_s^{t,x} \xi, h \rangle_H ds, \quad \tau \in [0, T], \quad \mathbb{P}\text{-a.s.} \quad (5.11)$$

Now we compute $\langle \tilde{Y}^{t,x}, W^\xi \rangle_\tau$ in a different way, using (5.24).

Let us define $\tilde{v}(t, x) = e^{-tA} v(t, x)$ so that we have $\tilde{v}(\tau, \Xi_\tau^{t,x}) = \tilde{Y}_\tau^{t,x}$. Moreover, we introduce the real function

$$\tilde{v}^h(\tau, x) = \langle \tilde{v}(\tau, x), h \rangle = \langle v(\tau, x), e^{-\tau(A^* + \lambda I)} h \rangle, \quad \tau \in [0, T], \quad x \in H.$$

By (5.7) we know that $\tilde{v}^h \in \mathcal{G}^{0,1}([0, T] \times H, \mathbb{R})$. Hence we can argue as in Lemma 6.3 of [17] (see also Lemma 6.4 in [17]) and obtain that the real process

$$(\tilde{v}^h(\tau, \Xi_\tau^{t,x}))_{\tau \in [0, T]} = (\langle \tilde{Y}_\tau^{t,x}, h \rangle)_{\tau \in [0, T]}$$

admits joint quadratic variation with W^ξ given by

$$\langle \tilde{v}^h(\cdot, \Xi_\cdot^{t,x}), W^\xi \rangle_\tau = \int_0^\tau \nabla^G \tilde{v}^h(s, \Xi_s^{t,x}) \xi ds = \int_0^\tau \langle \nabla^G v(s, \Xi_s^{t,x}) \xi, e^{-sA^*} h \rangle ds, \quad \tau \in [0, T].$$

Comparing this formula with (5.11) we discover that for $s \in [0, T]$, a.e., we have, \mathbb{P} -a.s.,

$$\langle e^{-sA} Z_s^{t,x} \xi, h \rangle = \langle e^{-sA} \nabla^G v(s, \Xi_s^{t,x}) \xi, h \rangle.$$

Thanks to the separability of H it follows that $e^{-sA} Z_s^{t,x} \xi = e^{-sA} \nabla^G v(s, \Xi_s^{t,x}) \xi$, for $s \in [0, T]$, a.e., \mathbb{P} -a.s.. The assertion follows easily. \square

We introduce now an approximation argument to smooth the coefficient B . We need such approximation in the proof of next theorem.

Recall that for $s \in [0, T]$, $B(s, \cdot) : H \rightarrow U$, and

$$GB(s, \cdot) = \begin{pmatrix} 0 \\ B(s, \cdot) \end{pmatrix}, \quad (5.12)$$

where B satisfies (5.2). To perform the approximations of B we follow [29]. For every $k \in \mathbb{N}$ we consider a nonnegative function $\rho_k \in C_b^\infty(\mathbb{R}^k)$ with compact support contained in the ball of radius $\frac{1}{k}$ and such that $\int_{\mathbb{R}^k} \rho_k(x) dx = 1$. Let $Q_k : H \rightarrow \langle g_1, \dots, g_k \rangle$ be the orthogonal projection on the linear space Λ_k generated by g_1, \dots, g_k , where $(g_k)_{k \geq 1}$ is a basis in H . We identify Λ_k with \mathbb{R}^k . For a bounded and continuous function $f : H \rightarrow U$ we set

$$f^k(x) = \int_{\mathbb{R}^k} \rho_k(y - Q_k x) f\left(\sum_{i=1}^k y_i g_i\right) dy,$$

where for every $k \in \mathbb{N}$, $y_k = \langle y, g_k \rangle_H$. It turns out that $f^k \in C_b^\infty(H, U)$.

We will apply this approximation to $f = B(s, \cdot)$, $s \in [0, T]$. By (5.2) it follows that the sequence $(B^k(t, \cdot))$ is equi-uniformly continuous on H , uniformly in $t \in [0, T]$, and $|B^k(t, x) - B(t, x)| \rightarrow 0$, as $k \rightarrow \infty$, for any $(t, x) \in [0, T] \times H$. Moreover, for $s \in [0, T]$, $B^n(s, \cdot)$ is Fréchet differentiable on H and

$$\sup_{(s, x) \in [0, T] \times H} \|\nabla B^n(s, x)\|_{L(H, U)} = c(n) \quad (5.13)$$

where $c(n) \rightarrow \infty$ as $n \rightarrow \infty$. For any $n \geq 1$ let us consider the FBSDE (5.1) with B^n in the place of B where again the precise meaning of the BSDE is given by its mild formulation:

$$\begin{aligned} Y_\tau^{n, t, x} &= \int_\tau^T e^{-(s-\tau)A} GB^n(s, \Xi_s^{t, x}) ds + \int_\tau^T e^{-(s-\tau)A} Z_s^{n, t, x} B^n(s, \Xi_s^{t, x}) ds \\ &\quad - \int_\tau^T e^{-(s-\tau)A} Z_s^{n, t, x} dW_s. \end{aligned} \quad (5.14)$$

We know that the map in (5.6) is Gâteaux differentiable on H , since in the BSDE (5.14) the coefficients are regular. Let $\xi \in U$ and consider the BSDE satisfied by the pair of processes $(\nabla_{G\xi} Y^{n,t,x}, \nabla_{G\xi} Z^{n,t,x})$, which can be obtained by differentiating (5.14) arguing as in [17], Proposition 4.8, or following [20], Proposition 4.4:

$$\begin{aligned} \nabla_{G\xi} Y_\tau^{n,t,x} &= \int_\tau^T e^{-(s-\tau)A} G \nabla B^n(s, \Xi_s^{t,x}) e^{(s-t)A} G \xi ds + \int_\tau^T e^{-(s-\tau)A} \nabla_{G\xi} Z_s^{n,t,x} B^n(s, \Xi_s^{t,x}) ds \\ &\quad + \int_\tau^T e^{-(s-\tau)A} Z_s^{n,t,x} \nabla B^n(s, \Xi_s^{t,x}) e^{(s-t)A} G \xi ds - \int_\tau^T e^{-(s-\tau)A} \nabla_{G\xi} Z_s^{n,t,x} dW_s. \end{aligned} \quad (5.15)$$

By applying estimate (5.4) to (5.15), and since $\|\nabla B^n(s, \cdot)\|_{L(H,U)} \leq c(n)$, where $c(n)$ does not depend on s and x , we get

$$\mathbb{E} \left[\sup_{\tau \in [0,T]} |\nabla_{G\xi} Y_\tau^{n,t,x}|^2 \right] + \mathbb{E} \int_0^T \|\nabla_{G\xi} Z_\tau^{n,t,x}\|_{L_2(U,H)}^2 d\tau \leq c(n, T)^2 |\xi|_U^2, \quad (5.16)$$

where $c(n, T) > 0$ is a constant that may blow up as $n \rightarrow \infty$, it depends on T and B but not on x and t . Recalling (5.7) we have

$$v^n(t, x) = Y_t^{n,t,x},$$

and we know that $v^n \in \mathcal{G}^{0,1}([0, T] \times H, H)$; by the previous computations we have, for any $n \geq 1$,

$$\sup_{t \in [0, T], x \in H} |\nabla_\xi^G v^n(t, x)| \leq c(n, T) |\xi|_U; \quad (5.17)$$

Let $\tau = t$ and let us take the expectation in (5.14); we get

$$Y_t^{n,t,x} = \mathbb{E} \int_t^T e^{-(s-t)A} G B^n(s, \Xi_s^{t,x}) ds + \mathbb{E} \int_t^T e^{-(s-t)A} Z_s^{n,t,x} B^n(s, \Xi_s^{t,x}) ds, \quad t \in [0, T], \quad x \in H. \quad (5.18)$$

Applying Lemma 5.3, we can write (5.18) as

$$v^n(t, x) = \mathbb{E} \int_t^T e^{-(s-t)A} G B^n(s, \Xi_s^{t,x}) ds + \mathbb{E} \int_t^T e^{-(s-t)A} \nabla^G v^n(s, \Xi_s^{t,x}) B^n(s, \Xi_s^{t,x}) ds, \quad (5.19)$$

for $t \in [0, T]$, $x \in H$. Using the H -valued OU transition semigroup (R_t) defined in (4.1), we obtain

$$v^n(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)A} G B^n(s, \cdot) + e^{-(s-t)A} \nabla^G v^n(s, \cdot) B^n(s, \cdot) \right] (x) ds, \quad t \in [0, T], \quad x \in H. \quad (5.20)$$

Now we want to show a convergence result of $Y_t^{n,t,x}$ to $Y_t^{t,x}$ and $Z^{n,t,x}$ to $Z^{t,x}$. The BSDE satisfied by $(Y^{n,t,x} - Y^{t,x}, Z^{n,t,x} - Z^{t,x})$ is

$$\begin{cases} -d(Y_\tau^{n,t,x} - Y_\tau^{t,x}) = -A(Y_\tau^{n,t,x} - Y_\tau^{t,x}) d\tau + (GB^n(\tau, \Xi_\tau^{t,x}) - GB(\tau, \Xi_\tau^{t,x})) d\tau \\ \quad + (Z_\tau^{n,t,x} B^n(\tau, \Xi_\tau^{t,x}) - Z_\tau^{t,x} B(\tau, \Xi_\tau^{t,x})) d\tau - (Z_\tau^{n,t,x} - Z_\tau^{t,x}) dW_\tau, \\ Y_T^{n,t,x} - Y_T^{t,x} = 0. \end{cases}$$

By adding and subtracting $Z_\tau^{t,x} B^n(\tau, \Xi_\tau^{t,x})$ this equation can be rewritten as

$$\begin{cases} -d(Y_\tau^{n,t,x} - Y_\tau^{t,x}) = -A(Y_\tau^{n,t,x} - Y_\tau^{t,x}) d\tau + (GB^n(\tau, \Xi_\tau^{t,x}) - GB(\tau, \Xi_\tau^{t,x})) d\tau \\ \quad - Z_\tau^{t,x} (B(\tau, \Xi_\tau^{t,x}) - B^n(\tau, \Xi_\tau^{t,x})) d\tau + (Z_\tau^{n,t,x} - Z_\tau^{t,x}) B^n(\tau, \Xi_\tau^{t,x}) d\tau - (Z_\tau^{n,t,x} - Z_\tau^{t,x}) dW_\tau, \\ Y_T^{n,t,x} - Y_T^{t,x} = 0. \end{cases} \quad (5.21)$$

Let $f_n(\tau) = (GB^n(\tau, \Xi_\tau^{t,x}) - GB(\tau, \Xi_\tau^{t,x})) - Z_\tau^{t,x} (B(\tau, \Xi_\tau^{t,x}) - B^n(\tau, \Xi_\tau^{t,x})) + (Z_\tau^{n,t,x} - Z_\tau^{t,x}) B^n(\tau, \Xi_\tau^{t,x})$. In the sequel we write $\|\cdot\|$ instead of $\|\cdot\|_{L_2(U,H)}$ to simplify notation. Let us fix $t \in [0, T]$. By Lemma 2.1 in [22] we know that there exists $M = M(A)$ such that, for any $\tau \in [0, T]$,

$$\mathbb{E} |Y_\tau^{n,t,x} - Y_\tau^{t,x}|^2 + \mathbb{E} \int_\tau^T \|Z_s^{n,t,x} - Z_s^{t,x}\|^2 ds \leq M(T - \tau) \mathbb{E} \int_\tau^T |f_n(s)|^2 ds. \quad (5.22)$$

Recall that the sequence (B^n) is uniformly bounded on $[0, T] \times H$ by $\|B\|_\infty$. Let $\eta > 0$ be small enough ($\eta M \|B\|_\infty \leq 1/2$). If $\tau \in [T - \eta, T]$ we get

$$\begin{aligned} \mathbb{E}|Y_\tau^{n,t,x} - Y_\tau^{t,x}|^2 + \frac{1}{2} \mathbb{E} \int_\tau^T \|Z_s^{n,t,x} - Z_s^{t,x}\|^2 ds &\leq M\eta \mathbb{E} \int_\tau^T |GB^n(s, \Xi_s^{t,x}) - GB(s, \Xi_s^{t,x})|^2 ds \\ &+ M\eta \mathbb{E} \int_\tau^T \|Z_s^{t,x}\|^2 |B(s, \Xi_s^{t,x}) - B^n(s, \Xi_s^{t,x})|_U^2 ds. \end{aligned}$$

Since $Z^{t,x} \in L^2_{\mathcal{P}}(\Omega \times [0, T], L_2(U, H))$, by the pointwise convergence of $B^n(\tau, \cdot)$ to $B(\tau, \cdot)$ and the dominated convergence theorem we get

$$\mathbb{E}|Y_\tau^{n,t,x} - Y_\tau^{t,x}|^2 + \mathbb{E} \int_\tau^T \|Z_s^{n,t,x} - Z_s^{t,x}\|^2 ds \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (5.23)$$

$\tau \in [T - \eta, T]$. Let now $\tau \in [(T - 2\eta) \vee 0, T - \eta]$. We consider (5.21) on $[0, T - \eta]$ with the terminal condition $Y_{T-\eta}^{n,t,x} - Y_{T-\eta}^{t,x} = 0$. Arguing as before we obtain (5.23) when $\tau \in [(T - 2\eta) \vee 0, T - \eta]$. Proceeding in this way we finally get (5.23) for any $\tau \in [0, T]$.

We mention two consequences of (5.23). The first one is obtained with $\tau = t$ and gives, setting $v^n(t, x) = Y_t^{n,t,x}$, $v^n(t, x) \rightarrow v(t, x)$ pointwise as $n \rightarrow \infty$ on $[0, T] \times H$. The second one is that, for $\tau \in [t, T]$, possibly passing to a subsequence, we can pass to the limit, \mathbb{P} -a.s., in $v^n(\tau, \Xi_\tau^{t,x}) = Y_\tau^{n,t,x}$ and get

$$v(\tau, \Xi_\tau^{t,x}) = Y_\tau^{t,x}, \quad \tau \in [t, T], \quad \mathbb{P} - a.s.. \quad (5.24)$$

Now we are ready to prove the following result.

Theorem 5.4. *Assume Hypothesis 1 and that B satisfy (5.2). Let v be the function defined in (5.7).*

Then $v \in B_b([0, T] \times H, H)$ and, for any $t \in [0, T]$, $v(t, \cdot) : H \rightarrow H$ admits the directional derivative $\nabla_{G\xi} v(t, x)$ in any $x \in H$ and along any direction $G\xi$, with $\xi \in U$ (see (2.4)). Moreover, for any $(t, x) \in [0, T] \times H$, the map: $\xi \mapsto \nabla_{G\xi} v(t, x) = \nabla_\xi^G v(t, x) \in L(U, H)$ and, for any $\xi \in U$, $\nabla_\xi^G v \in B_b([0, T] \times H, H)$ with $\sup_{(t,x) \in [0,T] \times H} \|\nabla^G v(t, x)\|_{L(U,H)} < \infty$. Finally, for any $\tau \in [0, T]$, a.e., we have

$$\nabla^G v(\tau, \Xi_\tau^{t,x}) = Z_\tau^{t,x}, \quad \mathbb{P} \text{ a.s..} \quad (5.25)$$

Proof. To prove the result we will pass to the limit as $n \rightarrow \infty$ in (5.20).

We first note that by estimate (5.23) $v^n(t, x) \rightarrow v(t, x)$ pointwise and so $v \in B_b([0, T] \times H, H)$.

Then we show that there exists $L : [0, T] \times H \rightarrow L(U, H)$ which is a bounded mapping, such that, for any $\xi \in U$, $L(t, x)\xi$ is measurable in $(t, x) \in [0, T] \times H$, and $\nabla^G v^n(t, x) \rightarrow L(t, x)$ in $L(U, H)$ pointwise as $n \rightarrow \infty$ on $[0, T] \times H$.

In order to obtain the assertion we start to study the difference $v^n(t, x) - v^{n+p}(t, x)$, $n, p \geq 1$. In the sequel we set $\mathcal{N}(0, Q_t) = \mu_t$. We have

$$\begin{aligned} v^n(t, x) - v^{n+p}(t, x) &= \int_t^T \int_H e^{-(s-t)A} \left[\nabla^G v^n(s, z + e^{(s-t)A}x) B^n(s, z + e^{(s-t)A}x) \right. \\ &\quad \left. - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x) B^{n+p}(s, z + e^{(s-t)A}x) \right] \mu_{s-t}(dz) ds \\ &+ \int_t^T \int_H e^{-(s-t)A} \left(GB^n(s, z + e^{(s-t)A}x) - GB^{n+p}(s, z + e^{(s-t)A}x) \right) \mu_{s-t}(dz). \end{aligned}$$

Since v^n and v^{n+p} are Gâteaux differentiable in the space variable, by the smoothing properties of the transition semigroup (R_s) , we can differentiate both sides and obtain for all $\xi \in U$ (cf. (4.4))

$$\begin{aligned} \nabla_\xi^G v^n(t, x) - \nabla_\xi^G v^{n+p}(t, x) &= \int_t^T \int_H e^{-(s-t)A} \left(GB^n(s, z + e^{(s-t)A}x) - GB^{n+p}(s, z + e^{(s-t)A}x) \right) \\ &\quad \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} G\xi, Q_{s-t}^{-1/2} z \right\rangle \mu_{s-t}(dz) ds \\ &+ \int_t^T \int_H e^{-(s-t)A} \left[\nabla^G v^n(s, z + e^{(s-t)A}x) B^n(s, z + e^{(s-t)A}x) \right. \\ &\quad \left. - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x) B^{n+p}(s, z + e^{(s-t)A}x) \right] \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} G\xi, Q_{s-t}^{-1/2} z \right\rangle \mu_{s-t}(dz) ds. \end{aligned}$$

Now in order to apply the Cauchy criterion we note that by (4.6)

$$\begin{aligned}
& \sup_{p \geq 1} \sup_{|\xi|_U=1} |\nabla_\xi^G v^n(t, x) - \nabla_\xi^G v^{n+p}(t, x)| \\
& \leq C \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H \sup_{p \geq 1} |GB^n(s, z + e^{(s-t)A}x) - GB^{n+p}(s, z + e^{(s-t)A}x)|^2 \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds \\
& + C \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H \sup_{p \geq 1} |\nabla^G v^n(s, z + e^{(s-t)A}x) B^n(s, z + e^{(s-t)A}x) \right. \\
& \quad \left. - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x) B^{n+p}(s, z + e^{(s-t)A}x)|^2 \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds = I_n + II_n
\end{aligned}$$

(to simplify notation we drop the dependence of I_n and II_n from (t, x)). We can easily apply the dominated convergence theorem, and letting $n \rightarrow \infty$ we get $I_n \rightarrow 0$. Concerning II_n , by adding and subtracting $\nabla^G v^n(s, z + e^{(s-t)A}x) B^{n+p}(s, z + e^{(s-t)A}x)$ we get

$$\begin{aligned}
II_n & \leq C \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H \sup_{p \geq 1} |\nabla^G v^n(s, z + e^{(s-t)A}x) [B^n(s, z + e^{(s-t)A}x) \right. \\
& \quad \left. - B^{n+p}(s, z + e^{(s-t)A}x)]|^2 \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds \\
& + C \sup_{p \geq 1} \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H |(\nabla^G v^n(s, z + e^{(s-t)A}x) - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x)) \right. \\
& \quad \left. B^{n+p}(s, z + e^{(s-t)A}x)|^2 \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds = II_n^a + II_n^b.
\end{aligned}$$

In order to estimate II_n^a we show that the sequence $(\nabla^G v^n)$ is equi-bounded from $[0, T] \times H$ with values in $L(U, H)$ (cf. (5.17)). We find, setting $\|\cdot\|_L = \|\cdot\|_{L(U, H)}$, with $\beta > 0$,

$$\begin{aligned}
& e^{\beta t} \|\nabla^G v^n(t, x)\|_L \\
& \leq C e^{\beta t} \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H |B^n(s, z + e^{(s-t)A}x)|_U^2 \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds \\
& + C e^{\beta t} \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H |\nabla^G v^n(s, z + e^{(s-t)A}x) B^n(s, z + e^{(s-t)A}x)|^2 \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds \\
& \leq C e^{\beta t} \|B\|_\infty \int_t^T (s-t)^{-\frac{1}{2}} ds + C \|B\|_\infty \int_t^T \frac{e^{-\beta(s-t)}}{(s-t)^{-\frac{1}{2}}} ds \cdot \sup_{t \in [0, T], y \in H} e^{\beta t} \|\nabla^G v^n(t, y)\|_L.
\end{aligned}$$

Since the sequence (B^n) is uniformly bounded, by taking the supremum over $(t, x) \in [0, T] \times H$ and β large enough we get on the left hand side we get

$$\sup_{n \geq 1} \sup_{t \in [0, T], y \in H} \|\nabla^G v^n(t, y)\|_{L(U, H)} < \infty. \quad (5.26)$$

Coming back to the estimate of II_n^a , we find

$$II_n^a \leq C_0 \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H \sup_{p \geq 1} |B^n(s, z + e^{(s-t)A}x) - B^{n+p}(s, z + e^{(s-t)A}x)|_U^2 \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds,$$

where C_0 is independent of t, x and n . By the dominated convergence theorem, using the pointwise convergence of the approximating sequence (B^n) , we find that $II_n^a \rightarrow 0$ as $n \rightarrow \infty$.

Concerning II_n^b , since (B_n) and $(\nabla^G v^n)$ are equi-bounded we have:

$$\begin{aligned}
& |B^{n+p}(s, z + e^{(s-t)A}x)|_U^2 \|\nabla^G v^n(s, z + e^{(s-t)A}x) - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x)\|_{L(U, H)}^2 \\
& \leq C_1 \|\nabla^G v^n(s, z + e^{(s-t)A}x) - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x)\|_{L(U, H)},
\end{aligned}$$

where C_1 is independent of t, x and n . Next, by the Hölder inequality,

$$\begin{aligned}
II_n^b &\leq c' \sup_{p \geq 1} \int_t^T (s-t)^{-\frac{1}{2}} \left(\int_H \|\nabla^G v^n(s, z + e^{(s-t)A}x) - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x)\|_L \mu_{s-t}(dz) \right)^{\frac{1}{2}} ds \\
&\leq c' \sup_{p \geq 1} \left(\int_t^T (s-t)^{-\frac{2}{3}} ds \right)^{\frac{3}{4}} \\
&\quad \cdot \left(\int_t^T \left(\int_H \|\nabla^G v^n(s, z + e^{(s-t)A}x) - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x)\|_L \mu_{s-t}(dz) \right)^2 ds \right)^{\frac{1}{4}} \\
&\leq c' \sup_{p \geq 1} \left(\int_t^T \int_H \|\nabla^G v^n(s, z + e^{(s-t)A}x) - \nabla^G v^{n+p}(s, z + e^{(s-t)A}x)\|_L^2 \mu_{s-t}(dz) ds \right)^{\frac{1}{4}} \\
&= c' \sup_{p \geq 1} \left(\mathbb{E} \int_t^T \|Z_s^{n,t,x} - Z_s^{n+p,t,x}\|^2 ds \right)^{\frac{1}{4}} \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$, for $(t, x) \in [0, T] \times H$, having applied (5.10) and estimate (5.23). Putting together these estimates we find

$$\sup_{p \geq 1} \|\nabla^G v^n(t, y) - \nabla^G v^{n+p}(t, y)\|_{L(U, H)} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (t, y) \in [0, T] \times H.$$

So we have proved that $\nabla^G v^n(t, x)$ converges as $n \rightarrow \infty$ in $L(U, H)$. We set

$$L(t, x) = \lim_{n \rightarrow \infty} \nabla^G v^n(t, x).$$

Clearly, for any $\xi \in U$, $\nabla_\xi^G v^n \in B_b([0, T] \times H, H)$ and so $(t, x) \mapsto L(t, x)\xi \in B_b([0, T] \times H, H)$.

Recall that v^n satisfies (5.20), so, by passing to the limit in (5.20), by the dominated convergence theorem, we get

$$v(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) + e^{-(s-t)A} L(s, \cdot) B(s, \cdot) \right] (x) ds. \quad (5.27)$$

By differentiating (5.20) in the direction $G\xi$, we get for all $\xi \in U$

$$\begin{aligned}
\nabla_\xi^G v^n(t, x) &= \int_t^T \int_H e^{-(s-t)A} GB^n(s, z + e^{(s-t)A}x) \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} G\xi, Q_{s-t}^{-1/2} z \right\rangle \mu_{s-t}(dz) \\
&\quad + \int_t^T \int_H e^{-(s-t)A} \nabla^G v^n(s, z + e^{(s-t)A}x) B^n(s, z + e^{(s-t)A}x) \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} G\xi, Q_{s-t}^{-1/2} z \right\rangle \mu_{s-t}(dz) ds.
\end{aligned}$$

By passing to the limit, we get

$$\begin{aligned}
L(t, x)\xi &= \int_t^T \int_H e^{-(s-t)A} GB(s, z + e^{(s-t)A}x) \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} G\xi, Q_{s-t}^{-1/2} z \right\rangle \mu_{s-t}(dz) \\
&\quad + \int_t^T \int_H e^{-(s-t)A} L(s, z + e^{(s-t)A}x) B(s, z + e^{(s-t)A}x) \left\langle Q_{s-t}^{-1/2} e^{(s-t)A} G\xi, Q_{s-t}^{-1/2} z \right\rangle \mu_{s-t}(dz) ds.
\end{aligned}$$

By considering (5.27) and taking into account the smoothing properties of the semigroup (R_t) (recall that, for any $s \in [0, T]$, $L(s, \cdot)B(s, \cdot) \in B_b(H, H)$, and Lemma 4.2 and (5.2)), we can easily obtain the desired differentiability of $v(t, \cdot) : H \rightarrow H$ along the directions $G\xi$, $\xi \in U$.

Taking the directional derivative in (5.27) we get, for all $\xi \in U$,

$$\nabla_\xi^G v(t, x) = \int_t^T \nabla_\xi^G R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) + e^{-(s-t)A} L(s, \cdot) B(s, \cdot) \right] (x) ds, \quad (t, x) \in [0, T] \times H,$$

and we finally deduce that $\nabla^G v(t, x) = L(t, x)$, $(t, x) \in [0, T] \times H$. By (5.23) we know that

$$Z^{n,t,x} \rightarrow Z^{t,x} \text{ in } L^2(\Omega \times [0, T]; L_2(U, H)).$$

Since, for any $n \geq 1$, $\tau \in [0, T]$, a.e., we have

$$\nabla^G v^n(\tau, \Xi_\tau^{t,x}) = Z_\tau^{n,t,x}, \quad \mathbb{P} \text{ a.s.},$$

we get easily that (5.25) holds. The proof is complete. \square

5.1 Additional regularity for the function $v(t, x) = Y_t^{t,x}$

Here we prove additional regularity properties for the function $v(t, x)$ defined in (5.7). By the representation formula given in (5.3) using the OU semigroup (R_t) we know that

$$v(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)A} GB(s, \cdot) \right] (x) ds + \int_t^T R_{s-t} \left[e^{-(s-t)A} \nabla^G v(s, \cdot) B(s, \cdot) \right] (x) ds. \quad (5.28)$$

Hence v satisfies an integral equation like (4.23) which has been studied in Theorem 4.5.

Lemma 5.5. *Let Hypotheses 1 and 2 hold true. Then the function v defined in (5.7) coincides with the function u , unique solution of (4.23) given in Theorem 4.5.*

Proof. By Theorem 5.4 we know that $v(t, x)$ belongs to $B_b([0, T] \times H, H)$ and, moreover, there exists $\nabla^G v : [0, T] \times H \rightarrow L(U, H)$ which is bounded and such that, for any $\xi \in U$, $\nabla_\xi^G v \in B_b([0, T] \times H, H)$.

We consider the difference between (4.23) and (5.28):

$$u(t, x) - v(t, x) = \int_t^T R_{s-t} \left[e^{-(s-t)A} (\nabla^G u(s, \cdot) B(s, \cdot) - \nabla^G v(s, \cdot) B(s, \cdot)) \right] (x) ds \quad (5.29)$$

and take the ∇^G -derivative:

$$\|\nabla^G u(t, x) - \nabla^G v(t, x)\|_{L(U, H)} = \left\| \int_t^T \nabla^G R_{s-t} \left[e^{-(s-t)A} (\nabla^G u(s, \cdot) B(s, \cdot) - \nabla^G v(s, \cdot) B(s, \cdot)) \right] (x) ds \right\|_L,$$

where $\|\cdot\|_L = \|\cdot\|_{L(U, H)}$. Since, $\nabla^G u(\cdot) B(\cdot)$ and $\nabla^G v(\cdot) B(\cdot)$ both belong to $B_b([0, T] \times H, H)$ we can apply Lemma 4.2 and obtain, for $\beta > 0$, $t \in [0, T]$,

$$\begin{aligned} & \sup_{x \in H} e^{\beta t} \|\nabla^G u(t, x) - \nabla^G v(t, x)\|_L \\ & \leq C \|B\|_\infty \int_t^T \frac{e^{-\beta(s-t)}}{(s-t)^{\frac{1}{2}}} ds \cdot \sup_{x \in H, s \in [0, T]} e^{\beta s} \|\nabla^G u(s, x) - \nabla^G v(s, x)\|_L \\ & \leq C_{\beta, T} \sup_{x \in H, s \in [0, T]} e^{\beta s} \|\nabla^G u(s, x) - \nabla^G v(s, x)\|_L, \quad t \in [0, T], \end{aligned}$$

where $C_{\beta, T} \rightarrow 0$ as $\beta \rightarrow \infty$. By choosing β large enough, we get $\sup_{x \in H, s \in [0, T]} \|\nabla^G u(s, x) - \nabla^G v(s, x)\|_L = 0$. So by (5.29) we get that u and v coincide. \square

6 Strong uniqueness for the wave equation

In this section we show how to remove the “bad” term B of equation (1.4), i.e.,

$$\begin{cases} dX_\tau^x = AX_\tau^x d\tau + GB(t, X_\tau^x) d\tau + GdW_\tau, & \tau \in [0, T], \\ X_0^x = x, \end{cases} \quad (6.1)$$

and get the main pathwise uniqueness result. Let $x \in H$. We consider a (weak) mild solution $(X_\tau^{t,x})_{\tau \in [0, T]}$ as in (2.7):

$$X_\tau^{t,x} = e^{(\tau-t)A} x + \int_t^\tau e^{(\tau-s)A} GB(s, X_s) ds + \int_t^\tau e^{(\tau-s)A} GdW_s, \quad \tau \in [t, T], \quad X_\tau^{t,x} = x, \quad \tau \leq t. \quad (6.2)$$

This in particular is a continuous H -valued process defined and adapted on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, on which it is defined a cylindrical U -valued \mathcal{F}_t -Wiener process W . Let us consider the FBSDE

$$\begin{cases} dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + GB(\tau, X_\tau^{t,x})d\tau + GdW_\tau, & \tau \in [t, T], \\ X_\tau^{t,x} = x, & \tau \in [0, t], \\ -d\tilde{Y}_\tau^{t,x} = -A\tilde{Y}_\tau^{t,x}d\tau + GB(\tau, X_\tau^{t,x})d\tau - \tilde{Z}_\tau^{t,x}dW_\tau, & \tau \in [0, T], \\ \tilde{Y}_T^{t,x} = 0. \end{cases} \quad (6.3)$$

The precise meaning of the BSDE in equation (6.3) is given by its mild formulation

$$\tilde{Y}_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{t,x}) ds - \int_\tau^T e^{-(s-\tau)A} \tilde{Z}_s^{t,x} dW_s, \quad \tau \in [0, T]. \quad (6.4)$$

Let us set

$$\tilde{W}_\tau = W_\tau + \int_0^\tau B(s, X_s^{t,x}) ds, \quad \tau \in [0, T].$$

By the Girsanov theorem, see e.g. [11] and [28], there exists a probability measure $\tilde{\mathbb{P}}$ such that on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \tilde{\mathbb{P}})$ the process (\tilde{W}_τ) is a cylindrical Wiener process in U up to time T . In the stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \tilde{\mathbb{P}})$ the FBSDE (6.3) can be rewritten as

$$\begin{cases} dX_\tau^{t,x} = AX_\tau^{t,x}d\tau + Gd\tilde{W}_\tau, & \tau \in [t, T], \\ X_\tau^{t,x} = x, & \tau \in [0, t], \\ -d\tilde{Y}_\tau^{t,x} = -A\tilde{Y}_\tau^{t,x}d\tau + GB(\tau, X_\tau^{t,x})d\tau + \tilde{Z}_\tau^{t,x}B(\tau, X_\tau^{t,x})d\tau - \tilde{Z}_\tau^{t,x}d\tilde{W}_\tau, & \tau \in [0, T], \\ \tilde{Y}_T^{t,x} = 0, \end{cases} \quad (6.5)$$

$$\tilde{Y}_\tau^{t,x} = \int_\tau^T e^{-(s-\tau)A} GB(s, X_s^{t,x}) ds + \int_\tau^T e^{-(s-\tau)A} \tilde{Z}_s^{t,x} B(s, X_s^{t,x}) ds - \int_\tau^T e^{-(s-\tau)A} \tilde{Z}_s^{t,x} d\tilde{W}_s, \quad (6.6)$$

$\tau \in [0, T]$. By the strong uniqueness for equation (2.5), $X^{t,x}$ is an Ornstein-Uhlenbeck process starting from x at t which is $\mathcal{F}_{t,T}^{\tilde{W}}$ -measurable (where $\mathcal{F}_{t,T}^{\tilde{W}}$ is the completed σ -algebra generated by $\tilde{W}_r - \tilde{W}_t$, $r \in [t, T]$). The law of $(X^{t,x}, \tilde{Y}^{t,x}, \tilde{Z}^{t,x})$ depends only on the coefficients of the FBSDE (6.5) and does not depend on the probability space on which it is defined the cylindrical Wiener process. Thus the law of $(X^{t,x}, \tilde{Y}^{t,x}, \tilde{Z}^{t,x})$ coincides with the one of $(\Xi^{t,x}, Y^{t,x}, Z^{t,x})$ solution of the FBSDE (5.1).

Moreover $Y_t^{t,x}$ and $\tilde{Y}_t^{t,x}$ are both deterministic and so they define a unique function $v(t, x)$ given in (5.7). Moreover, we have, for any $\tau \in [0, T]$,

$$\tilde{Y}_\tau^{t,x} = v(\tau, X_\tau^{t,x}), \quad \mathbb{P} - a.s.; \quad \text{for any } \tau \in [0, T] \text{ a.e., } \tilde{Z}_\tau^{t,x} = \nabla^G v(\tau, X_\tau^{t,x}), \quad \mathbb{P} - a.s. \quad (6.7)$$

(cf. (5.24) and (5.25)). In order to prove strong existence of a mild solution to equation (6.1), we will rewrite in a different way (6.2), removing the term $\int_t^\tau e^{(t-s)A} GB(s, X_s) ds$ by means of the BSDE in (6.6); when $t = 0$ we denote, for brevity, by $(\tilde{Y}^x, \tilde{Z}^x)$ the process $(\tilde{Y}^{0,x}, \tilde{Z}^{0,x})$.

Proposition 6.1. *Let Hypotheses 1 and 2 holds true. Then a (weak) mild solution $X^x = (X_\tau^x)$ of (6.2) starting at $t = 0$ satisfies, for any $\tau \in [0, T]$, \mathbb{P} -a.s.,*

$$X_\tau^x = e^{\tau A} x + e^{\tau A} v(0, x) - v(\tau, X_\tau^x) + \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) dW_s + \int_0^\tau e^{(\tau-s)A} G dW_s. \quad (6.8)$$

Proof. Let us fix $\tau \in [0, T]$. Writing (6.4) for $t = 0$ and $\tau = 0$ we find, \mathbb{P} -a.s.,

$$\begin{aligned} v(0, x) = \tilde{Y}_0^x &= \int_0^T e^{-sA} GB(s, X_s^x) ds - \int_0^T e^{-sA} \tilde{Z}_s^x dW_s \\ &= \int_0^\tau e^{-sA} GB(s, X_s^x) ds - \int_0^\tau e^{-sA} \tilde{Z}_s^x dW_s + \int_\tau^T e^{-sA} GB(s, X_s^x) ds - \int_\tau^T e^{-sA} \tilde{Z}_s^x dW_s. \end{aligned} \quad (6.9)$$

In (6.4) with $t = 0$ we apply to both sides the bounded linear operator $e^{-\tau A}$, we get

$$e^{-\tau A} \tilde{Y}_\tau^x = \int_\tau^T e^{-sA} GB(s, X_s^x) ds - \int_\tau^T e^{-sA} \tilde{Z}_s^x dW_s, \quad \tau \in [0, T]. \quad (6.10)$$

Using (6.7) we obtain, \mathbb{P} -a.s.,

$$\begin{aligned} v(0, x) &= e^{-\tau A} \tilde{Y}_\tau^x + \int_0^\tau e^{-sA} GB(s, X_s^x) ds - \int_0^\tau e^{-sA} \tilde{Z}_s^x dW_s \\ &= e^{-\tau A} v(\tau, X_\tau^x) + \int_0^\tau e^{-sA} GB(s, X_s^x) ds - \int_0^\tau e^{-sA} \nabla^G v(s, X_s^x) dW_s. \end{aligned} \quad (6.11)$$

In particular from (6.11) we get

$$\int_0^\tau e^{-sA} GB(s, X_s^x) ds = v(0, x) - e^{-\tau A} v(\tau, X_\tau^x) + \int_0^\tau e^{-sA} \nabla^G v(s, X_s^x) dW_s \quad (6.12)$$

and by applying the bounded linear operator $e^{\tau A}$ to both sides we deduce that, \mathbb{P} -a.s.,

$$\int_0^\tau e^{(\tau-s)A} GB(s, X_s^x) ds = e^{\tau A} v(0, x) - v(\tau, X_\tau^x) + \int_0^\tau e^{(\tau-s)A} \nabla^G v(s, X_s^x) dW_s. \quad (6.13)$$

Since $X_\tau^x - e^{\tau A} x - \int_0^\tau e^{(\tau-s)A} G dW_s = \int_0^\tau e^{(\tau-s)A} GB(s, X_s) ds$ we get (6.8). \square

Remark 6.2. Notice that formula (6.8) does not coincide with formula (7) in [7], which is obtained by the so-called Itô-Tanaka trick. In fact our function v (see 5.7) and the function U used in the paper [7] are different, and we can see this by comparing (5.28) in the present paper with the mild formula (16) in [7]. Following the procedure in [7], one should consider $U : [0, T] \times H \rightarrow H$ represented by the real functions $U_n := \langle U, e_n \rangle : [0, T] \times H \rightarrow \mathbb{R}$, where $(e_n)_{n \geq 1}$ is a basis in H , and U_n is the solution to the linear Kolmogorov equation

$$\begin{cases} \frac{\partial U_n(t, x)}{\partial t} + \mathcal{L}_t[U_n(t, \cdot)](x) = -GB_n(t, x), & x \in H, \quad t \in [0, T], \\ U_n(T, x) = 0. \end{cases} \quad (6.14)$$

where $\mathcal{L}_t[f](x) = \frac{1}{2} \text{Tr}(GG^* \nabla^2 f(x)) + \langle Ax, \nabla f(x) \rangle + \langle GB(t, x), \nabla f(x) \rangle$; one can solve (6.14) with techniques similar to the ones used also in [19]. On the other hand, from (5.28) we formally see that v is an H -valued solution of the following equation which contains the operator A :

$$\begin{cases} \frac{\partial v(t, x)}{\partial t} + \mathcal{L}_t[v(t, \cdot)](x) = Av(t, x) - GB(t, x), & x \in H, \quad t \in [0, T], \\ v(T, x) = 0. \end{cases} \quad (6.15)$$

Theorem 6.3. *Let Hypotheses 1 and 2 hold true. Then for equation (1.4) pathwise uniqueness holds. Moreover, there exists $c_T > 0$ such that if $X_\tau^{x_1}$ and $X_\tau^{x_2}$ are two (weak) mild solutions starting from x_1 and x_2 at $t = 0$ (defined on the same stochastic basis) then*

$$\sup_{\tau \in [0, T]} \mathbb{E} |X_\tau^{x_1} - X_\tau^{x_2}|^2 \leq c_T |x_1 - x_2|^2, \quad x_1, x_2 \in H. \quad (6.16)$$

Proof. We prove (6.16) which implies the pathwise uniqueness. Let us fix $x_1, x_2 \in H$ and consider two (weak) mild solutions X^1 and X^2 defined on the same stochastic basis, with respect to the same cylindrical Wiener process and starting respectively from x_1 and x_2 at time $t = 0$. Let $T_0 \in (0, T]$ be such that $h(T_0) \cdot (\sup_{t \in [0, T]} \|B(t, \cdot)\|_\alpha) \leq 1/4$ (see (4.27) in Theorem 4.5).

We consider the FBSDE (5.1) with $T = T_0$. We find the function $v^{(0)} : [0, T_0] \times H \rightarrow H$ according to (5.7) with $T = T_0$. By (6.8) we know that

$$\begin{aligned} X_\tau^1 - X_\tau^2 &= e^{\tau A} (x_1 - x_2) + e^{\tau A} [v^{(0)}(0, x_1) - v^{(0)}(0, x_2)] \\ &\quad - [v^{(0)}(\tau, X_\tau^1) - v^{(0)}(\tau, X_\tau^2)] + \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)] dW_s, \quad \tau \in [0, T_0]. \end{aligned} \quad (6.17)$$

By the regularity properties of $v^{(0)}$, see Theorem 4.5, formula (4.27) and Lemma 5.5, we get

$$\begin{aligned} & |e^{\tau A}(x_1 - x_2)| + |e^{\tau A}[v^{(0)}(0, x_1) - v^{(0)}(0, x_2)]| + |v^{(0)}(\tau, X_\tau^1) - v^{(0)}(\tau, X_\tau^2)| \\ & \leq C|x_1 - x_2| + \frac{1}{3}|X_\tau^1 - X_\tau^2|, \quad \tau \in [0, T_0]. \end{aligned}$$

Concerning the stochastic integral, by the Itô isometry (see [11], Section 4.3) we find

$$\begin{aligned} & \mathbb{E} \left| \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)] dW_s \right|^2 \\ & \leq \mathbb{E} \int_0^\tau \|\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)\|_{L_2(U, H)}^2 ds. \end{aligned}$$

Let us consider a basis (e_k) in U ; by the regularity properties of $v^{(0)}$ get

$$\begin{aligned} & \mathbb{E} \int_0^\tau \|\nabla^G v^{(0)}(s, X_s^1) - \nabla^G v^{(0)}(s, X_s^2)\|_{L_2(U, H)}^2 ds = \sum_{j \geq 1} \mathbb{E} \int_0^\tau |\nabla_{e_j}^G v^{(0)}(s, X_s^1) - \nabla_{e_j}^G v^{(0)}(s, X_s^2)|^2 ds \\ & = \sum_{j \geq 1} \mathbb{E} \int_0^\tau ds \left| \int_0^1 \nabla \nabla_{e_j}^G v^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1)) [X_s^2 - X_s^1] dr \right|^2 \\ & \leq \int_0^1 dr \int_0^\tau \mathbb{E} \sum_{j \geq 1} |\nabla \nabla_{e_j}^G v^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1)) [X_s^2 - X_s^1]|^2 ds \\ & = \int_0^1 dr \int_0^\tau \mathbb{E} \|\nabla_{X_s^2 - X_s^1} \nabla^G v^{(0)}(s, X_s^1 + r(X_s^2 - X_s^1))\|_{L_2(U, H)}^2 ds \\ & \leq \sup_{(t, x) \in [0, T_0] \times H} \sup_{|h|=1} \|\nabla_h \nabla^G v^{(0)}(t, x)\|_{L_2(U, H)}^2 \int_0^\tau \mathbb{E} |X_s^1 - X_s^2|^2 ds. \end{aligned}$$

Starting from (6.17) and using the previous estimates, we can apply the Gronwall lemma and obtain

$$\sup_{\tau \in [0, T_0]} \mathbb{E} |X_\tau^{x_1} - X_\tau^{x_2}|^2 \leq c_T |x_1 - x_2|^2. \quad (6.18)$$

If $T_0 < T$ we consider the FBSDE (5.1) with terminal time $(2T_0) \wedge T$. We find $v^{(1)} : [0, (2T_0) \wedge T] \times H \rightarrow H$ according to (5.28) with T replaced by $(2T_0) \wedge T$. By (6.8) we obtain in particular

$$\begin{aligned} & X_\tau^1 - X_\tau^2 = e^{\tau A}(x_1 - x_2) + e^{\tau A}[v^{(1)}(0, x_1) - v^{(1)}(0, x_2)] \\ & - [v^{(1)}(\tau, X_\tau^1) - v^{(1)}(\tau, X_\tau^2)] + \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(1)}(s, X_s^1) - \nabla^G v^{(1)}(s, X_s^2)] dW_s, \quad \tau \in [T_0, (2T_0) \wedge T]. \end{aligned}$$

By Theorem 4.5, formula (4.27) and Lemma 5.5, we get, for $\tau \in [T_0, (2T_0) \wedge T]$,

$$\begin{aligned} & |e^{\tau A}(x_1 - x_2)| + |e^{\tau A}[v^{(1)}(0, x_1) - v^{(1)}(0, x_2)]| + |v^{(1)}(\tau, X_\tau^1) - v^{(1)}(\tau, X_\tau^2)| \\ & \leq C|x_1 - x_2| + \frac{1}{3}|X_\tau^1 - X_\tau^2|. \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \left| \int_0^\tau e^{(\tau-s)A} [\nabla^G v^{(1)}(s, X_s^1) - \nabla^G v^{(1)}(s, X_s^2)] dW_s \right|^2 \\ & \leq \sup_{(t, x) \in [0, (2T_0) \wedge T] \times H} \sup_{|h|=1} \|\nabla_h \nabla^G v^{(1)}(t, x)\|_{L_2(U, H)}^2 \int_0^\tau \mathbb{E} |X_s^1 - X_s^2|^2 ds \\ & = \sup_{(t, x) \in [0, (2T_0) \wedge T] \times H} \sup_{|h|=1} \|\nabla_h \nabla^G v^{(1)}(t, x)\|_{L_2(U, H)}^2 \left(\int_0^{T_0} \mathbb{E} |X_s^1 - X_s^2|^2 ds + \int_{T_0}^\tau \mathbb{E} |X_s^1 - X_s^2|^2 ds \right) \\ & \leq \sup_{(t, x) \in [0, (2T_0) \wedge T] \times H} \sup_{|h|=1} \|\nabla_h \nabla^G v^{(1)}(t, x)\|_{L_2(U, H)}^2 (c_T T_0 |x_1 - x_2|^2 + \int_{T_0}^\tau \mathbb{E} |X_s^1 - X_s^2|^2 ds). \end{aligned}$$

We have obtained, for $\tau \in [T_0, (2T_0) \wedge T]$,

$$\mathbb{E}|X_\tau^1 - X_\tau^2|^2 \leq C_T |x_1 - x_2|^2 + \int_{T_0}^\tau \mathbb{E}|X_s^1 - X_s^2|^2 ds.$$

By the Gronwall lemma we find $\sup_{\tau \in [T_0, (2T_0) \wedge T]} \mathbb{E}|X_\tau^{x_1} - X_\tau^{x_2}|^2 \leq c_T |x_1 - x_2|^2$. Proceeding in this way, in finite steps, we get (6.16). \square

Applying the previous theorem together with an infinite dimensional generalization of the Yamada-Watanabe theorem (see Theorem 2 in [28] and Appendix in [24]) we obtain (see also Remark 2.1):

Corollary 6.4. *Assume the same hypotheses of Theorem 6.3. Then, for any $x \in H$, there exists a strong mild solution to (1.4).*

A Appendix: an estimate on the minimal control energy for the controlled wave equation

We consider a positive self-adjoint operator S on a real separable Hilbert space K , i.e., $S : \mathcal{D}(S) \subset K \rightarrow K$. Note that here the compactness of S^{-1} is dispensed with. We introduce the Hilbert space

$$M = \mathcal{D}(S^{\frac{1}{2}}) \times K$$

endowed with the inner product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle_M = \left\langle S^{\frac{1}{2}} y_1, S^{\frac{1}{2}} y_2 \right\rangle_K + \langle z_1, z_2 \rangle_K.$$

We also introduce

$$\mathcal{D}(A) = \mathcal{D}(S) \times \mathcal{D}(S^{\frac{1}{2}}), \quad A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -S & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \text{for every } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A). \quad (\text{A.1})$$

The operator A is the generator of the contractive group e^{tA} on M

$$e^{tA} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} \cos \sqrt{S}t & \frac{1}{\sqrt{S}} \sin \sqrt{S}t \\ -\sqrt{S} \sin \sqrt{S}t & \cos \sqrt{S}t \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad t \in \mathbb{R}.$$

We consider the following linear controlled system in M :

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = k \in M, \end{cases} \quad (\text{A.2})$$

where $G : K \rightarrow M$ is defined by $Gu = \begin{pmatrix} 0 \\ u \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} u$ and the control $u \in L_{loc}^2(0, \infty; K)$. We remark that $\text{Im } G = \left\{ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \in M : a_1 = 0 \right\}$.

It is well known that equation (A.2) is null controllable for any $t > 0$ and any initial state in M , see e.g. [6], pp. 149 and pp.153. This is equivalent to say that, for any $t > 0$,

$$e^{tA}(M) \subset Q_t^{1/2}(M), \quad \text{where } Q_t = \int_0^t e^{sA} G G^* e^{sA^*} ds. \quad (\text{A.3})$$

(cf. Section 2). Moreover the minimal energy $\mathcal{E}_C(t, k)$ steering a general initial state $k = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ to 0 in time t behaves like $t^{-\frac{3}{2}} |k|_M$ as t goes to 0 (see e.g. [1] for a more general result). Recall that $\mathcal{E}_C(t, k)$ is the infimum of

$$\left(\int_0^t |u(r)|_K^2 dr \right)^{1/2}$$

over all controls $u \in L^2(0, t; K)$ driving the solution w from k to 0 in time t . It can be proved that

$$\mathcal{E}_C(t, k) = |Q_t^{-1/2} e^{tA} k|_M$$

(see, for instance, [34]). Hence if $\mathcal{E}_C(t) = \sup_{|k|_M=1} \mathcal{E}_C(t, k)$, we know that

$$\mathcal{E}_C(t) \text{ is } O(t^{-3/2}), \text{ as } t \rightarrow 0^+. \quad (\text{A.4})$$

On the other hand, we have the following estimate for the minimal energy steering an initial state $k \in \text{Im}(G)$ to 0 at time t . A similar result has been proved in [25] by a spectral approach in the case of the wave equation in $H_0^1([0, 1]) \times L^2([0, 1])$.

Theorem A.1. *There exists a positive constant C such that, for any $k = \begin{pmatrix} 0 \\ a \end{pmatrix} \in \text{Im}(G)$,*

$$|\mathcal{E}_C(t, k)| \leq \frac{C|k|_M}{t^{\frac{1}{2}}} = \frac{C|a|_K}{t^{\frac{1}{2}}}, \quad t > 0, \quad (\text{A.5})$$

Proof. The proof below is inspired by [30] (see also [34] page 19). We consider the Hilbert space

$$H = K \times \mathcal{D}(S^{-\frac{1}{2}})$$

(by $\mathcal{D}(S^{-\frac{1}{2}})$ we mean the completion of K with respect to the norm $|S^{-1/2} \cdot|$; this is a Hilbert space; see also Section 2) endowed with the inner product

$$\left\langle \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} \right\rangle_H = \left\langle S^{-\frac{1}{2}} z_1, S^{-\frac{1}{2}} z_2 \right\rangle_K + \langle y_1, y_2 \rangle_K. \quad (\text{A.6})$$

We also consider an extension of the unbounded operator A given in (4.10) which we still denote by A :

$$\mathcal{D}(A) = \mathcal{D}(S^{1/2}) \times K, \quad A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -S & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -Sy \end{pmatrix} \in H, \quad \text{for every } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A).$$

Clearly A generates a contractive group e^{tA} on M and moreover if $a \in K$ we have

$$k = \begin{pmatrix} 0 \\ a \end{pmatrix} \in \mathcal{D}(A) \quad \text{and} \quad e^{tA} \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{S}} \sin(\sqrt{S}t) a \\ \cos(\sqrt{S}t) a \end{pmatrix} \in \mathcal{D}(S^{1/2}) \times K, \quad t \in \mathbb{R}. \quad (\text{A.7})$$

Let us fix $T > 0$ and $k = \begin{pmatrix} 0 \\ a \end{pmatrix}$ with $a \in K$. Consider $f(t) = t^2(T - t)^2$ and

$$\phi(t) = \frac{f(t)}{\int_0^T f(s) ds}, \quad t \in [0, T].$$

Note that $\phi(0) = \phi(T)$, $\int_0^T \phi(s) ds = 1$ and there exists $C > 0$ (independent of $T > 0$) such that $|\phi(t)| \leq \frac{C}{T}$ and $|\phi'(t)| \leq \frac{C}{T^2}$, $t \in [0, T]$. Let $\psi : [0, T] \rightarrow H$,

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix} = -\phi(t) e^{tA} k = -\begin{pmatrix} \phi(t) \frac{1}{\sqrt{S}} \sin(\sqrt{S}t) a \\ \phi(t) \cos(\sqrt{S}t) a \end{pmatrix}, \quad t \in [0, T].$$

Using also the derivative ψ'_1 we introduce the control

$$u(t) = \psi_2(t) + \psi'_1(t) \in K, \quad t \in [0, T]. \quad (\text{A.8})$$

We show that it transfers k to 0 at time T . We have

$$\int_0^T e^{(T-s)A} G u(s) ds = \int_0^T e^{(T-s)A} \begin{pmatrix} 0 \\ \psi_2(s) \end{pmatrix} ds + \int_0^T e^{(T-s)A} G \psi'_1(s) ds.$$

Since $G\psi_1'(s) = \begin{pmatrix} 0 \\ \psi_1'(s) \end{pmatrix}$ is continuous from $[0, T]$ with values in $\mathcal{D}(A)$ (cf. (A.7)) integrating by parts we get

$$\begin{aligned} \int_0^T e^{(T-s)A} G\psi_1'(s) ds &= \int_0^T e^{(T-s)A} A G\psi_1(s) ds = \int_0^T e^{(T-s)A} \begin{pmatrix} 0 & I \\ -S & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \psi_1(s) \end{pmatrix} ds \\ &= \int_0^T e^{(T-s)A} \begin{pmatrix} \psi_1(s) \\ 0 \end{pmatrix} ds. \end{aligned}$$

Hence we find

$$\int_0^T e^{(T-s)A} G u(s) ds = - \int_0^T \phi(s) e^{(T-s)A} e^{sA} k ds = -e^{TA} k.$$

Now we compute the energy of the control u : $\int_0^T |u(s)|^2 ds$. First note that

$$\int_0^T |\psi_2(t)|_K^2 dt = \int_0^T \phi(t)^2 |\cos(\sqrt{S}t)a|_K^2 dt \leq \frac{|a|_K^2}{T}.$$

On the other hand using the spectral theorem for self-adjoint operators we get

$$\begin{aligned} \int_0^T |\psi_1(t)'|_K^2 dt &= \int_0^T \left| \phi(t) \cos(\sqrt{S}t)a + \phi'(s) \frac{1}{\sqrt{S}} \sin(\sqrt{S}t) a \right|_K^2 dt \\ &\leq \frac{2|a|_K^2}{T} + 2|a|_K^2 \int_0^T \left| \phi'(t)t \frac{1}{\sqrt{S}t} \sin(\sqrt{S}t) \right|_K^2 dt \leq \frac{c|a|_K^2}{T}, \end{aligned}$$

where c is independent on T . Collecting the previous estimates we obtain

$$\mathcal{E}_C(T, k) \leq \left(\int_0^T |u(s)|_K^2 ds \right)^{1/2} \leq \frac{C|k|_K}{\sqrt{T}}, \quad T > 0. \quad \square$$

Now let U be a real separable Hilbert space and let $\Lambda : \mathcal{D}(\Lambda) \subset U \rightarrow U$ be a positive self-adjoint operator on U as in Section 4.6. We also consider the Hilbert space

$$V = \mathcal{D}(\Lambda^{1/2})$$

and its dual space V' which can be identified with the completion of U with respect to the norm $|\Lambda^{-1/2} \cdot|_U$ (see Section 3.4 in [31]). The operator Λ can be extended to a positive self-adjoint operator on V' which we still denote by Λ with domain V :

$$\Lambda : V \subset V' \rightarrow V'. \quad (\text{A.9})$$

It turns out that the square root of such extension has domain $U \subset V'$ (the proof of this fact is simple in our case since Λ has a diagonal form; recall that we assume that Λ^{-1} is of trace class).

We need to apply Theorem A.1 in the case when $K = V'$ and $S = \Lambda$. Moreover

$$M = H = U \times V',$$

$$\mathcal{D}(A) = V \times U, \quad A \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 & I \\ -\Lambda & 0 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}, \quad \text{for every } \begin{pmatrix} y \\ z \end{pmatrix} \in \mathcal{D}(A)$$

(cf. (A.1)). The operator $G : V' \rightarrow H$ is defined by $Ga = \begin{pmatrix} 0 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ I \end{pmatrix} a$, $a \in V'$. The associated controlled system is

$$\begin{cases} \dot{w}(t) = Aw(t) + Gu(t), \\ w(0) = h \in H, \end{cases} \quad (\text{A.10})$$

with $u \in L_{loc}^2(0, \infty; V')$. By (A.4) and Theorem A.1 we get

Corollary A.2. *Let us consider the minimal energy*

$$\mathcal{E}_C(t, h) = |Q_t^{-1/2} e^{tA} h|_H, \quad h \in H,$$

for (A.10). We have, for $t \in (0, T)$

$$\begin{aligned} |Q_t^{-1/2} e^{tA} h|_H &\leq \frac{cT}{t^{3/2}} |h|_H \\ |Q_t^{-1/2} e^{tA} Ga|_H &\leq \frac{c}{t^{1/2}} |Ga|_H = \frac{c}{t^{1/2}} |a|_{V'}, \quad h \in H, \quad a \in V'. \end{aligned}$$

In particular, since $U \subset V'$,

$$|Q_t^{-1/2} e^{tA} Ga|_H \leq \frac{c}{t^{1/2}} |Ga|_H = \frac{c}{t^{1/2}} |\Lambda^{-1/2} a|_U, \quad a \in U. \quad (\text{A.11})$$

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